Session 87:

Innovations in Teaching Astronomy

87.25

BRINGING PULSATING STARS

INTO THE

PHYSICS CLASSROOM

Bradley W. Carroll

and

Dale A. Ostlie

Department of Physics

Weber State University

Ogden, UT

Abstract

Undergraduate physics students at many institutions are never exposed to any aspect of theoretical astrophysics. This is doubly unfortunate because students then miss the opportunity to integrate their knowledge of several disciplines and apply it to interesting astrophysical phenomena. Students find pulsating stars inherently interesting, even more so when the role of these stars in determining cosmological distances is explained. Delta Cephei is such a star, with a pulsation period of 5 days, 10 hours, 48 minutes. Introductory physics students can combine the equations of gravitation and adiabatic gas processes with Newton's 2nd law to produce a simple numerical model of Delta Cephei that precisely reproduces its period. Graphs of the oscillations are distinctly nonsinusoidal. Student curiosity about where the period is "hiding" in the equations can lead to the technique of linearization and a discussion of how small departures from stable equilibrium result in simple harmonic motion. This paper will be available at physics.weber.edu/carroll/ on the web.

The Problem:

Students in calculus-based introductory physics courses rarely see different areas of physics brought together to solve an interesting problem. In their eyes,

$$\text{Newton's Laws} = \text{projectile motion}$$

$$\text{gravitation} = \text{free fall acceleration}$$

$$\text{thermodynamics} = \text{heat engines}$$

$$\text{simple harmonic motion} = \text{mass on a spring}$$

These students *never* see astrophysical applications, with the possible exception of Kepler's laws.

Astrophysics is the melting pot of physical theory, where diverse areas of physics are brought together to answer some of the most interesting questions in science.
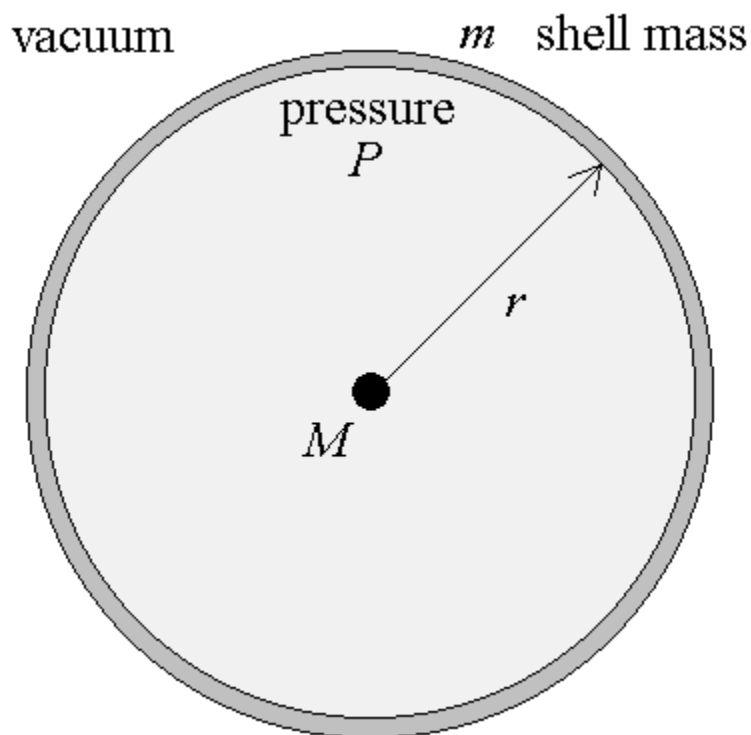
One Answer:

Students in our calculus-based introductory physics courses find Cepheid variable stars to be inherently interesting. We have assigned students the task of calculating a simple model of a variable star. This problem leads to several interesting questions about the physics that governs these objects.

The Assignment:

(as given to our students)

Cepheid variable stars are pulsating red giant stars that regularly expand and contract in a rhythmic manner. Astronomers compare observations of these stars with the results of detailed computer models in order to test their theories of stellar structure and stellar evolution. In this exercise, you will make a very simple model of a Cepheid variable star that will still reveal some of the interesting features of these stars.

A star is a massive sphere of hot gas. For your model star, suppose that the entire mass M of the star is concentrated in a point at the center, and that the surface layers of mass m $(m \ll M)$ are represented by a thin spherical shell of radius r. This shell is filled with a massless gas of pressure P. (The pressure is the same everywhere inside the shell.) Outside the shell is a vacuum.

vacuum　　　　　　　　　　m　shell mass

pressure
P

r

M

When the model star is in equilibrium, the outward force of gas pressure on the shell exactly balances the inward gravitational pull of the central mass M on the shell. However, if the radius of the shell is larger than its equilibrium radius, the lessened gas pressure will allow the shell to fall inward. As it falls, the shell compresses the gas; the increased gas pressure halts the compression, and pushes the shell back outward in an expansion. Note that the pressure and radius both change as the model star pulsates.

The equations that describe the oscillation of the model star are Newton's 2nd Law for the forces on the shell:

$$m\frac{dv}{dt} = 4\pi r^2 P - \frac{GMm}{r^2} \qquad (1)$$

and the definition of velocity:

$$v = \frac{dr}{dt}. \qquad (2)$$

A common assumption used to study pulsating stars is that the expansion and contraction of the gas is *adiabatic*:

$$P_i V_i^\gamma = P_f V_f^\gamma, \qquad (3)$$

where $V = \frac{4}{3}\pi r^3$ is the volume of the spherical shell of gas, and γ is the ratio of specific heats. The "i" and "f" subscripts identify the initial and final values of the pressure and volume.

1. **Explain in words the meaning of each term in Eq. (1). Also, use Eq. (3) to show that**

$$P_i r_i^{3\gamma} = P_f r_f^{3\gamma}. \tag{4}$$

2. **You will not be taking derivatives. Instead, you will take the difference between the initial and final values of the radius r and radial velocity v of the shell divided by the time interval Δt separating the initial and final values. That is, you will use**

$$\frac{v_f - v_i}{\Delta t} \quad \text{instead of} \quad \frac{dv}{dt},$$

and

$$\frac{r_f - r_i}{\Delta t} \quad \text{instead of} \quad \frac{dr}{dt}$$

in Eqs. (1) and (2).

A careful analysis shows that you should set $r = r_i$ and $P = P_i$ on the right-hand side of Eq. (1), and set $v = v_f$ on the left-hand side of Eq. (2). Make these substitutions in Eqs. (1) and (2), and show that you can write

$$v_f = v_i + \left(\frac{4\pi r_i^2 P_i}{m} - \frac{GM}{r_i^2} \right) \Delta t \tag{5}$$

and

$$r_f = r_i + v_f \Delta t. \tag{6}$$

These equations, together with

$$P_f = P_i \left(\frac{r_i}{r_f} \right)^3, \tag{7}$$

will determine the pulsation of your model star.

3. **Now you can calculate the oscillation of the model star. The mass of a typical Cepheid variable star is**

$$M = 1 \times 10^{31} \text{ kg,}$$

and the mass of the surface layers may taken as

$$m = 1 \times 10^{26} \text{ kg.}$$

For starting values at time $t = 0$, take

$$r_i = 1.7 \times 10^{10} \text{ m, } v_i = 0 \text{ m s}^{-1}, \text{ } P_i = 5.6 \times 10^4 \text{ N m}^{-2}$$

and use a time interval of $\Delta t = 10^4$ s. Take the ratio of specific heats, γ, to be the value for an ideal monatomic gas, $\gamma = 5/3$.

Use Eq. (5) to calculate the final velocity v_f at the end of one time interval (at time $t = 1 \times 10^4$ s); then use Eq. (6) to calculate the final radius r_f and Eq. (7) to calculate the final pressure P_f. Now take these final values to be your new initial values, and find new values for r, v, and P after two time intervals (at time $t = 2 \times 10^4$ s). Continue on for 150 time intervals, until $t = 1.5 \times 10^6$ s. Make three graphs of your results: r vs. t, v vs. t, and P vs. t. Plot the time on the horizontal axis.

4. From your graphs, measure the period T of the oscillation (both in seconds and in days) and the average radius of the model star, R. Compare this value of the period with the formula (found using more advanced techniques)

$$T = \frac{2\pi}{\sqrt{(3\gamma - 4)\frac{GM}{R^3}}}. \tag{8}$$

Real Cepheid variable stars have periods that range between 1 and 50 days, with characteristic values between 5 and 10 days. The first Cepheid variable star was discovered in 1784 by John Goodricke, and is named Delta Cephei. You may want to compare your results with the those for Delta Cephei: its pulsation period is 5 days, 8 hours, and 48 minutes, and its surface expands and contracts with a maximum radial velocity of about 20 km s^{-1}.

— — — — — — —

Note that this assignment asks students to pull together their knowledge of

- Newtonian mechanics

- gravitation

- thermodynamics

- simple harmonic motion

As an example, here are the results at the end of the first time interval ($t = 1 \times 10^4$ s). Using the initial values of $r_i = 1.7 \times 10^{10}$ m, $v_i = 0$ m s^{-1}, and $P_i = 5.6 \times 10^4$ N m^{-2}, Eq. (5) gives the new velocity

$$v_f = v_i + \left(\frac{4\pi r_i^2 P_i}{m} - \frac{GM}{r_i^2} \right) \Delta t$$

$$= 0 + \left(\frac{4\pi (1.7 \times 10^{10})^2 (5.6 \times 10^4)}{m} \right) \Delta t$$

$$- \left(\frac{GM}{(1.7 \times 10^{10})^2} \right) \Delta t$$

$$= -2.7491 \times 10^3 \text{ m s}^{-1}.$$

This value of v_f is plugged into Eq. (6) to find the new radius

$$r_f = r_i + v_f \Delta t$$

$$= 1.7 \times 10^{10} + (-2.7491 \times 10^3) \Delta t$$

$$= 1.6973 \times 10^{10} \text{ m}.$$

Finally, this value of r_f is used in Eq. (7) to find the new pressure

$$P_f = P_i \left(\frac{r_i}{r_f} \right)^{3\gamma}$$

$$= (5.6 \times 10^4) \left(\frac{1.7 \times 10^{10}}{1.6973 \times 10^{10}} \right)^{3\gamma}$$

$$= 5.6455 \times 10^4 \text{ N m}^{-2}.$$

Now we have the values v_f, r_f, and P_f at the end of the first time interval ($t = 1 \times 10^4$ sec). Next, set $v_i = v_f$, $r_i = r_f$, and $P_i = P_f$, and repeat the above procedure to find the radial velocity, radius, and pressure of the star at the end of the second time interval($t = 2 \times 10^4$ s), and so on for all 150 time intervals.

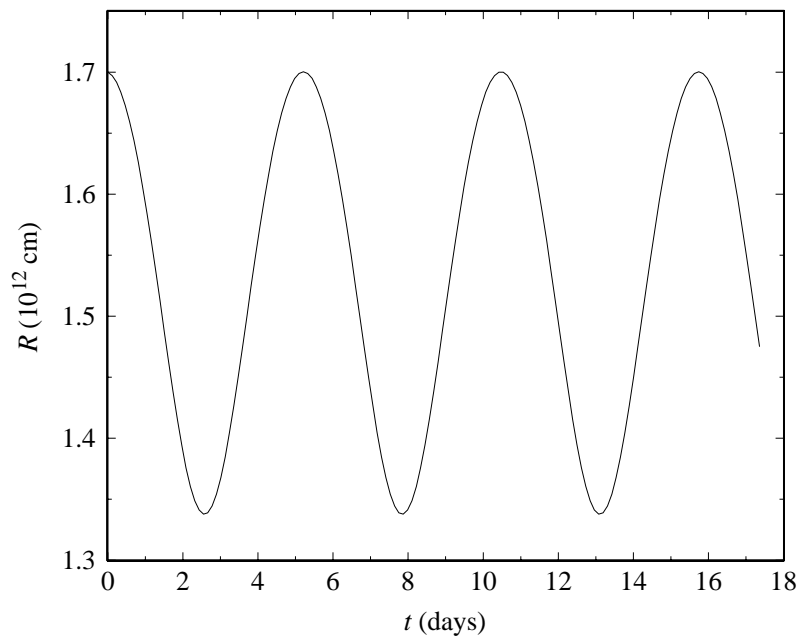The results have been used to produce the following graphs.
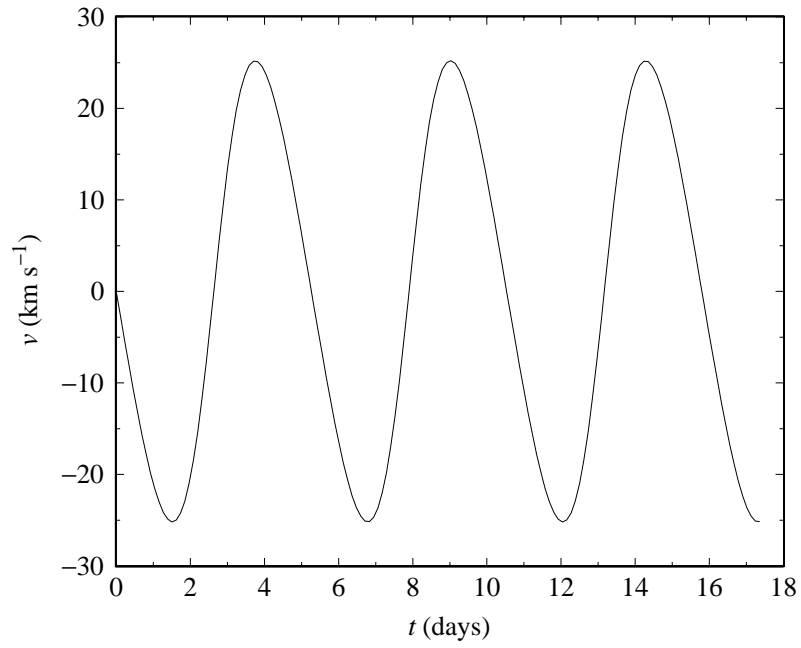

Figure 1: Graph of stellar radius vs. time.
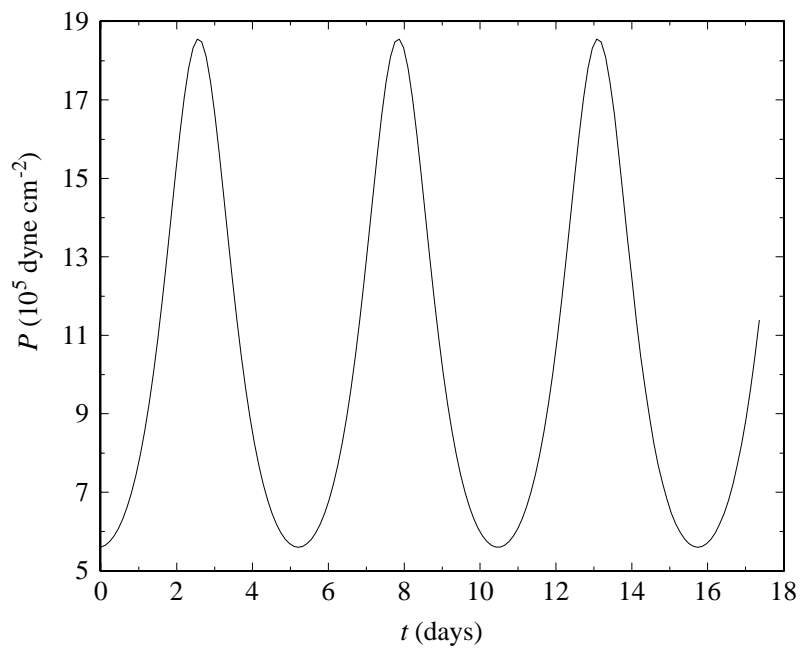
Figure 2: Graph of surface velocity vs. time.

Figure 3: Graph of pressure vs. time.

Note that the graphs are not perfectly sinusoidal, especially P vs. t in Figure 3. (This is a *nonlinear* calculation.

From the results of the calculations and the graphs, the period of the oscillation is $T = 5.25$ d $= 4.53 \times 10^5$ s, and the equilibrium radius is $R_\circ = 1.52 \times 10^{12}$ cm.

Equation (8) for the period of the *linearized* one-zone model gives $T_{\mathrm{lin}} = 4.56 \times 10^5$ s $= 5.28$ d, in good agreement with the nonlinear period.

Confession:

The student may be skeptical about the close agreement of the theoretical and observed periods. (In fact, the equilibrium radius of the model star was chosen to produce a result close to δ Cephei's measured period of $T_{\mathrm{obs}} = 5^{\mathrm{d}}8^{\mathrm{h}}48^{\mathrm{m}} = 5.37$ d.)

Skeptical Student:

"Give me a break. What happens if you change one of the numbers. How about a different value for m, the mass of the shell?"

Response:

"Try it and see for yourself!"

The surprised student finds that changing the value of m does not change the period of the one-zone model!

Skeptical Student:

"OK — I see that there is no m in Eq. (8). But where did that equation come from? Where is that formula hiding in Eqs. (1) − (3)?"

Response:

"To dig Eq. (8) out of our equations, we must *linearize* them. Here's how to get started ...."

A Linearized One-Zone Model

We will write every variable in the differential equations as an equilibrium value (found in the static model of the star) plus a small change due to the pulsation. (This means that the result will only be valid for small-amplitude oscillations.) For example, the pressure P would be written as $P = P_\circ + \delta P$, where $P_\circ$ is the value of the pressure within a mass shell of the equilibrium model, and δP is the small change in pressure that occurs as that mass shell moves in the oscillating model star. Thus δP is a function of time, but $P_\circ$ is constant. When the variables written in this manner are inserted into the differential equations we started with, Eqs. (1) $-$ (3), the terms containing only equilibrium quantities cancel and terms that involve powers of the deltas higher than the first, such as $(\delta P)^2$, may be discarded because they are negligibly small.

We used Newton's second law (Eq. 1) applied to the shell,

$$m\frac{d^2R}{dt^2} = -\frac{GMm}{R^2} + 4\pi R^2 P. \qquad (9)$$

For the equilibrium model, the left-hand side of this equation is zero, so

$$\frac{GMm}{R_\circ^2} = 4\pi R_\circ^2 P_\circ. \qquad (10)$$

The linearization is accomplished by writing the star's radius and pressure as

$$R = R_\circ + \delta R \qquad \text{and} \qquad P = P_\circ + \delta P,$$

and inserting these expressions into Eq. (9), giving

$$m\frac{d^2(R_\circ + \delta R)}{dt^2} = -\frac{GMm}{(R_\circ + \delta R)^2} + 4\pi(R_\circ + \delta R)^2(P_\circ + \delta P). \quad (11)$$

Using the first-order approximation

$$\frac{1}{(R_\circ + \delta R)^2} \approx \frac{1}{R_\circ^2}\left(1 - 2\frac{\delta R}{R_\circ}\right)$$

and keeping only those terms involving the first powers of the deltas results in

$$\begin{aligned} m\frac{d^2(\delta R)}{dt^2} &= -\frac{GMm}{R_\circ^2} + \frac{2GMm}{R_\circ^3}\delta R \\ &\quad + 4\pi R_\circ^2 P_\circ + 8\pi R_\circ P_\circ \delta R + 4\pi R_\circ^2 \delta P, \end{aligned} \quad (12)$$

where $d^2 R_\circ/dt^2 = 0$ has been used for the equilibrium model. The first and third terms on the right-hand side cancel (see Eq. 10), leaving

$$m\frac{d^2(\delta R)}{dt^2} = \frac{2GMm}{R_\circ^3}\delta R + 8\pi R_\circ P_\circ \delta R + 4\pi R_\circ^2 \delta P. \quad (13)$$

This is the linearized version of Newton's second law for our one-zone model.

We must also linearize Eq. (3), the adiabatic relation $PV^\gamma = \text{constant}$, where γ is the ratio of specific heats of the gas. Since the volume of the one-zone model is just $\frac{4}{3}\pi R^3$, the adiabatic relation says that $PR^{3\gamma} = \text{constant}$. The linearized version of this is

$$\frac{\delta P}{P_\circ} = -3\gamma\frac{\delta R}{R_\circ}. \quad (14)$$

Using this equation, δP can be eliminated from Eq. (13). In addition, Eq. (10) can be used to replace $8\pi R_\circ P_\circ$ with

$2GMm/R_\circ^3$. **As a result, the mass m of the shell cancels, leaving the linearized equation for δR:**

$$\frac{d^2(\delta R)}{dt^2} = -(3\gamma - 4)\frac{GM}{R_\circ^3}\,\delta R. \tag{15}$$

If $\gamma > 4/3$ (so the right-hand side of the equation is negative), this is just the familiar equation for simple harmonic motion. It has the solution $\delta R = A\sin(\omega t)$, where A is the pulsation amplitude and ω is the angular pulsation frequency. Inserting this expression for δR into Eq. (15) results in

$$\omega^2 = (3\gamma - 4)\frac{GM}{R_\circ^3}. \tag{16}$$

Finally, the pulsation period of the one-zone model is just $T = 2\pi/\omega$, which is Eq. (8),

$$T = \frac{2\pi}{\sqrt{(3\gamma - 4)\frac{GM}{R^3}}}.$$

Skeptical Student:

"Wow, that was a lot of work! But if there is an equation for the period, why not just use it? Why did we have to go through a big numerical calculation?"

Response:

"Did you notice how the pulsation amplitude, A, canceled in this example? You can't calculate the amplitude of the oscillations with a linearized approach to stellar pulsation."

Skeptical Student:

"I'm still amazed that the equations we started with [Eqs. (1) – (3)] contained simple harmonic motion. Were we just lucky?"

Response:

"Not at all. Most systems in nature will undergo simple harmonic motion if you displace them slightly from equilibrium."

Skeptical Student:

"You mean a star works the same way as a mass on a spring? I don't believe it!"

Response:

"Fortunately, science works even if you don't believe in it! Let's see why simple harmonic motion is so common."

Suppose we have a general potential energy function, $U(r)$, for a force $\mathbf{F} = -\left(dU/dr\right)\hat{\mathbf{r}}$ on a particle of mass m, and suppose that $r = 0$ is a stable equilibrium point. Expanding $U(r)$ in a Taylor series about the origin results in

$$U(r) = U(0) + \left.\frac{dU}{dr}\right|_0 r + \frac{1}{2}\left.\frac{d^2U}{dr^2}\right|_0 r^2 + \cdots,$$

where the higher-order terms may be neglected for small displacements from the origin. [These are precisely the terms that were thrown away when we linearized Eqs. (1 − (3).] Potential energies are defined only to within an additive constant, so you can always choose the constant so $U(0) = 0$. Also, because the origin is a stable equilibrium point, $dU/dr = 0$ at the origin and $d^2U/dr^2 > 0$ there. The force on the particle is therefore

$$\mathbf{F} = -\frac{dU(r)}{dr}\,\hat{\mathbf{r}} \simeq -\left.\frac{d^2U}{dr^2}\right|_0 r\,\hat{\mathbf{r}} = -\left.\frac{d^2U}{dr^2}\right|_0 \mathbf{r},$$
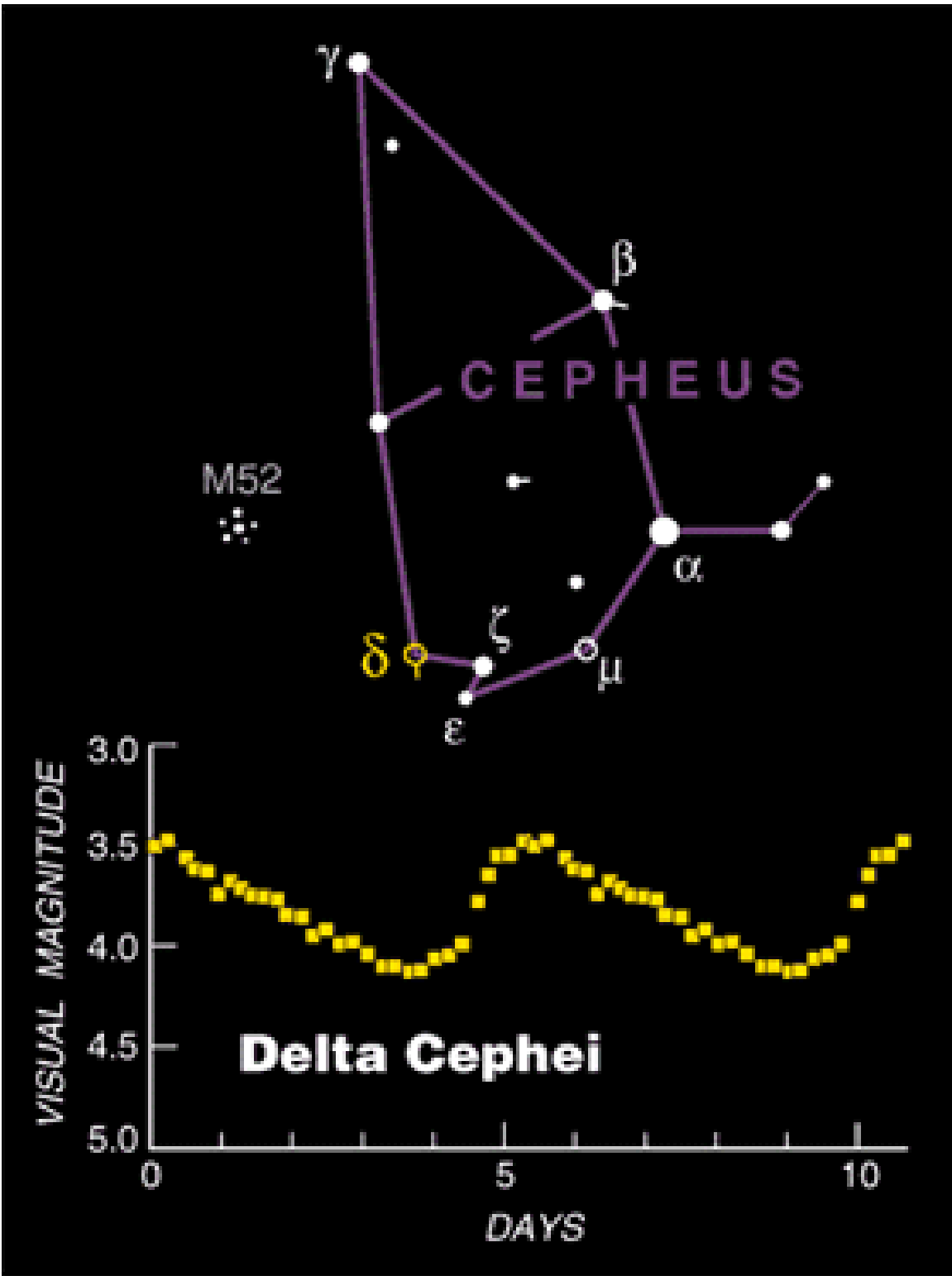
which is in the form of Hooke's law for a spring. Therefore if the particle is displaced slightly from the origin and then released, it will undergo simple harmonic motion about the origin.

When Eqs. (1) − (3) were used to numerically solve for the motion of the pulsating mass shell, nothing was thrown away. In that case, the oscillations were *non*linear, and the graphs were not perfectly sinusoidal. The actual light curve of the star is decidedly nonlinear.

Enthusiastic Student:

"I'm going outside tonight to see this for myself!"

The Light Curve of Delta Cephei

Source: "The Top 12 Naked- Eye Variable Stars" by John Isles, at
http://www.skypub.com/sights/variables/top12var.html