I. BASIC ASPECTS OF STELLAR STRUCTURE AND PULSATION

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1. Introduction

The purpose of this chapter is to provide a general background for the more specific chapters to follow. Much of the application of variability in astrophysics is concerned with stellar structure; thus we briefly review the calculation of stellar structure and evolution. Furthermore, we present the 'machinery' of stellar pulsations. Throughout, the emphasis is on general principles, rather than on specific applications. It is evident that the exposition will be far from complete. For more detailed background information on stellar evolution one of the many books on the subject may be consulted (*e.g.* Kippenhahn & Weigert 1990; Hansen & Kawaler 1994; Cox & Giuli 1968); much more extensive presentations on stellar pulsations were provided by, for example, Unno *et al.* (1989), Cox (1980), Gough (1993), and Gautschy & Saio (1995, 1996). Also, Christensen-Dalsgaard *et al.* (1999) (in the following Chapter II) present applications to the helioseismic studies of the solar interior; further details on the crucially important treatment of the equation of state and opacity are provided by Däppen & Guzik (1999) (in the following Chapter III).

The observable properties of stellar pulsations depend on the underlying stellar structure and dynamics and hence, in principle, all provide potential tools for probing stellar interiors. The practical possibilities for using these tools depend both on the ease and precision with which a given pulsation property can be observed, and the extent and certainty with which it can be related to aspects of the stellar interior. Stellar oscillation frequencies can be determined observationally with extremely high accuracy. Furthermore, the frequencies are related in a relatively simple way to the structure and rotation of the stars; in many cases, linear analysis, often even assuming the adiabatic approximation, is adequate, making the frequencies very direct measures of stellar interiors. The processes that excite the oscillations are also fairly well understood for several types of pulsating stars; thus the limits of the regions of instability, say, can be related to the physics of the stellar interiors. In addition, it is sometimes possible to measure the damping rates of modes, which may provide more detailed information about the physical processes causing the damping. The effects that control the limiting amplitude of the oscillations are in general rather more uncertain, although in a few cases information is emerging from the amplitude distribution. Finally, phase differences and amplitude ratios between different observables, for a given mode, reflect the behaviour of the oscillation in, and hence may provide information about, the properties of the atmosphere.

Stellar pulsations can be excited in two fundamentally different ways: through self-excitation or by an external force. In the former case, there are regions where the pulsation operates as a heat engine, extracting mechanical energy from the energy flow through the star; even though in other regions the tendency is for the motion to be dissipated, the net result is that mode is linearly unstable. This driving is typically associated with specific features in the opacity, and it requires that the location of these features satisfies certain constraints. As a result, the instability is typically restricted to fairly well-defined regions of stellar parameters; a typical example is the Cepheid instability strip. Also, the excitation may be selective, operating only for modes in a fairly restricted frequency band. It is important to note that no linear stability calculation provides information about the limiting amplitude of the modes; this must result from nonlinear interactions, presumably involving either the saturation of the driving mechanism or interactions with other modes. However, it is fair to say that these issues are still far from being understood.

Other stars, including the Sun, are observed to pulsate even though linear stability calculations indicate that the relevant modes are damped. In this case, the oscillations require forcing, the most likely source being turbulent convection in the near-surface regions of the star. Since stellar convection has a broad spectrum of timescales, such forcing is expected to lead to the excitation of modes over a range of frequencies, although of course depending on the properties of the modes. Also, in this case the oscillation amplitudes may be estimated from the balance between the energy input from the forcing and the damping; in fact, in the solar case such estimates are not inconsistent with the observed amplitudes.

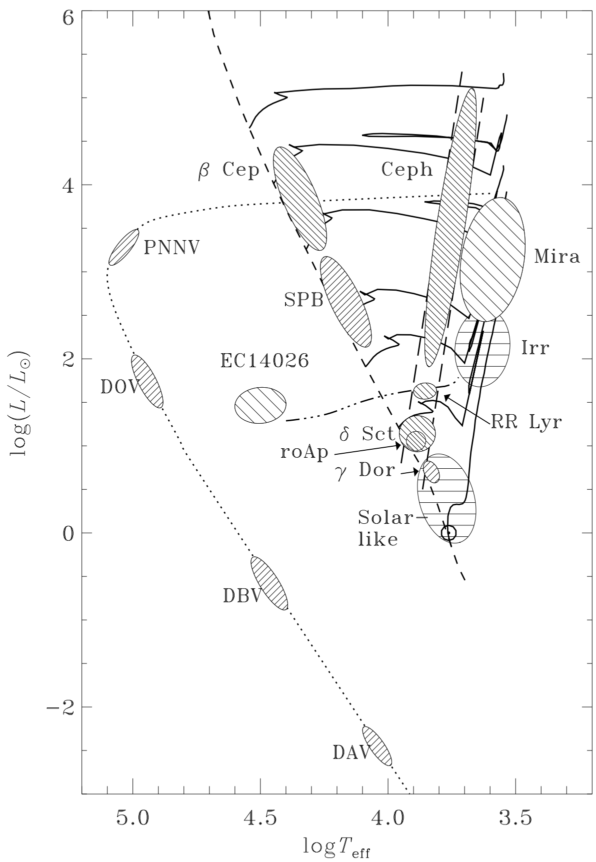

Figure 1. Schematic Hertzsprung-Russell diagram illustrating the location of several classes of pulsating stars. The dashed line shows the zero-age main sequence, the continuous curves are selected evolution tracks, at masses 1, 2, 3, 4, 7, 12 and 20 $M_\odot$, the dot-dashed line is the horizontal branch and the dotted curve is the white-dwarf cooling curve.

As a small taste of the riches available through the analysis of pulsating stars, Fig. 1 shows schematically the location of pulsating stars in the Hertzsprung-Russell diagram. Many of these are discussed in considerable detail in these proceedings. It is striking that pulsation has been found, or is suspected, for stars of virtually all types. Although much remains still to be done before we can use observations of these pulsations to investigate the detailed stellar properties in all these cases, the observations obviously have the potential for very extensive tests of the theory of stellar structure and evolution.

2. Equations of stellar evolution and pulsation

2.1. GENERAL EQUATIONS OF HYDRODYNAMICS

To provide a background for the treatment of stellar pulsations, we give a brief summary of the basic equations of hydrodynamics. For more detailed treatments, any of the large number of basic textbooks may be consulted (*e.g.* Landau & Lifshitz 1966; Batchelor 1967).

A hydrodynamical system is characterized by specifying the physical quantities as functions of position $\mathbf{r}$ and time t. These properties include the local density $\rho(\mathbf{r}, t)$, the local pressure $p(\mathbf{r}, t)$, and any other thermodynamic quantity that may be needed, as well as the local instantaneous velocity $\mathbf{v}(\mathbf{r}, t)$. Here $\mathbf{r}$ denotes the position vector to a given point in space, and the description therefore corresponds to what is seen by a stationary observer. This is known as the so-called *Eulerian* description. In addition, we shall also use the so-called *Lagrangian* description, following the motion of a given parcel of fluid. To these descriptions correspond the time derivative $\partial/\partial t$ seen by a stationary observer, and the derivative $\mathrm{d}/\mathrm{d}t$ observed when following the motion; the latter is also known as the material (or Stokes) time derivative. The local velocity is obviously determined by the rate of change of the position $\mathbf{r}$ of a fluid parcel:

$$\mathbf{v}(\mathbf{r}, t) = \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} \; . \tag{1}$$

This may furthermore be used to relate the two time derivatives of some quantity ϕ:

$$\frac{\mathrm{d}\phi}{\mathrm{d}t} = \left(\frac{\partial \phi}{\partial t}\right)_{\mathbf{r}} + \boldsymbol{\nabla}\phi \cdot \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} = \frac{\partial \phi}{\partial t} + \mathbf{v} \cdot \boldsymbol{\nabla}\phi \, . \tag{2}$$

Conservation of mass is expressed by *the equation of continuity*:

$$\frac{\partial \rho}{\partial t} + \mathrm{div}\,(\rho \mathbf{v}) = 0 \; . \tag{3}$$

With the aid of equation (2), equation (3) may also be written

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} + \rho \,\mathrm{div}\,\mathbf{v} = 0 \; , \tag{4}$$

thus giving the rate of change of density in a given parcel of flowing gas. If we define the *specific volume* as $V_\rho = 1/\rho$, which measures the volume taken up by a unit of mass, then an alternative formulation of equation (4) is

$$\frac{1}{V_\rho}\frac{\mathrm{d}V_\rho}{\mathrm{d}t} = \mathrm{div}\,\mathbf{v}\,. \tag{5}$$

Hence $\mathrm{div}\,\mathbf{v}$ is the rate of expansion of a given volume of gas during its motion.

Under stellar conditions we can generally ignore internal friction (or *viscosity*) in the gas. The *equation of motion*, per unit volume, can then be written

$$\rho\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = -\boldsymbol{\nabla} p + \rho\,\mathbf{f}\,, \tag{6}$$

where $\mathbf{f}$ is the body force per unit mass, which has yet to be specified. Here the first term on the right-hand side is the surface force, given by the pressure p. Combining equations (2) and (6) yields an alternative form of the equation of motion (also known as the *Euler equations*),

$$\rho\frac{\partial\mathbf{v}}{\partial t} + \rho\mathbf{v}\cdot\boldsymbol{\nabla}\mathbf{v} = -\boldsymbol{\nabla} p + \rho\,\mathbf{f}\,. \tag{7}$$

Among the possible body forces we here consider only gravity. Thus, in particular, we neglect effects of magnetic fields. The force per unit mass from gravity is the gravitational acceleration $\mathbf{g}$, which can be written as the gradient of the gravitational potential Φ:

$$\mathbf{g} = \boldsymbol{\nabla}\Phi\,, \tag{8}$$

where Φ satisfies Poisson's equation

$$\nabla^2\Phi = -4\pi G\rho\,, \tag{9}$$

G being the gravitational constant.

To complete the description, we need to relate p and ρ. This is done through the energetics of the system, as described by the first law of thermodynamics. By applying it to a volume of unit mass, moving with the fluid, we obtain *the energy equation*

$$\frac{\mathrm{d}Q}{\mathrm{d}t} = \frac{\mathrm{d}E}{\mathrm{d}t} + p\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{\rho}\right) = \frac{\mathrm{d}E}{\mathrm{d}t} - \frac{p}{\rho^2}\frac{\mathrm{d}\rho}{\mathrm{d}t} = \frac{\mathrm{d}E}{\mathrm{d}t} + \frac{p}{\rho}\,\mathrm{div}\,\mathbf{v}\,. \tag{10}$$

Here $\mathrm{d}Q/\mathrm{d}t$ is the rate of heat loss or gain per unit mass of material, and E the internal energy per unit mass. The equation may be cast in more convenient forms by using thermodynamic identities (*e.g.* Cox & Giuli 1968):

$$\frac{\mathrm{d}Q}{\mathrm{d}t} = \frac{1}{\rho(\Gamma_3 - 1)} \left(\frac{\mathrm{d}p}{\mathrm{d}t} - \frac{\Gamma_1 p}{\rho} \frac{\mathrm{d}\rho}{\mathrm{d}t} \right) \tag{11}$$

$$= c_P \left(\frac{\mathrm{d}T}{\mathrm{d}t} - \frac{\Gamma_2 - 1}{\Gamma_2} \frac{T}{p} \frac{\mathrm{d}p}{\mathrm{d}t} \right) \tag{12}$$

$$= c_V \left[\frac{\mathrm{d}T}{\mathrm{d}t} - (\Gamma_3 - 1) \frac{T}{\rho} \frac{\mathrm{d}\rho}{\mathrm{d}t} \right] . \tag{13}$$

Here T is temperature, c_P and c_V are the specific heats per unit mass at constant pressure and volume, and the *adiabatic exponents* are defined by

$$\Gamma_1 = \left(\frac{\partial \ln p}{\partial \ln \rho} \right)_{\mathrm{ad}} , \quad \frac{\Gamma_2 - 1}{\Gamma_2} = \left(\frac{\partial \ln T}{\partial \ln p} \right)_{\mathrm{ad}} ,$$

$$\Gamma_3 - 1 = \left(\frac{\partial \ln T}{\partial \ln \rho} \right)_{\mathrm{ad}} , \tag{14}$$

the derivatives being at constant specific entropy, *i.e.*, corresponding to an adiabatic change.

The heating term in equations (10) – (13) can be written as

$$\rho \frac{\mathrm{d}Q}{\mathrm{d}t} = \rho\varepsilon - \mathrm{div}\, \boldsymbol{\mathcal{F}}. \tag{15}$$

Here ε is the rate of energy generation per unit mass (from, for example, thermonuclear reactions), and $\boldsymbol{\mathcal{F}}$ is the flux of energy. In general, radiation is the dominant contribution to $\boldsymbol{\mathcal{F}}$ in stars. Here we consider just the diffusion approximation to radiative transport, according to which the radiative flux is given by

$$\boldsymbol{\mathcal{F}}_{\mathrm{rad}} = - \frac{4a\tilde{c}T^3}{3\kappa\rho} \boldsymbol{\nabla} T , \tag{16}$$

where κ is the opacity, $\tilde{c}$ the speed of light, and a is the radiation density constant. This provides a relation between the state of the gas and the radiative flux, which is analogous to a simple conduction equation. It might be noted that in the interiors of highly evolved stars, including white dwarfs, heat conduction by degenerate electrons is important; this can formally be included in equation (16) by suitably modifying the opacity (*e.g.* Kippenhahn & Weigert 1990).

The relative importance of the left-hand and the right-hand side in equation (15) depends on the relevant time scales. During normal stellar evolution, conditions change so slowly that $\mathrm{d}Q/\mathrm{d}t$ can sometimes be neglected compared with, for example, $\mathrm{div}\, \boldsymbol{\mathcal{F}}$ [see also eq. (25) below]. On the other hand, for rapidly varying phenomena such as stellar pulsations, each term in $\mathrm{d}Q/\mathrm{d}t$ is often very much larger than the right-hand side; as will be discussed in more detail in Section 2.3.2, this leads to *adiabatic approximation* where the right-hand side of equation (15) is neglected. In that case, for example, we obtain from equation (11) that

$$\frac{\mathrm{d}p}{\mathrm{d}t} = \frac{\Gamma_1 p}{\rho} \frac{\mathrm{d}\rho}{\mathrm{d}t} , \tag{17}$$

a very simple relation between p and ρ.

At a microscopic level, equation (16) provides a complete description of the flux of energy in stellar interiors. However, often transport by turbulent gas motion must be taken into account. This is the case in convection zones, where rising hot gas and descending cool gas dominate the energy transport. Ideally, the entire hydrodynamical system, including convection, must be described as a whole. However, the resulting equations are too complex to be handled analytically or numerically, in general treatments of stellar structure or pulsation. Thus it is customary to separate out the convective motions by performing averages of the equations over length scales that are large compared with the convective motion, but small compared with other scales of interest. In this case the convective flux appears as an additional contribution in equation (15); it must be determined from the other quantities characterizing the system along with consideration of the equations for the turbulent motion. This is generally done quite crudely; a familiar example is the mixing-length theory.

Full description of a hydrodynamical system also requires information about the properties of matter in the system. In particular, we need an equation of state, defining the relation between pressure, density and temperature as well as the thermodynamical quantities appearing in the energy equation, equations (11) – (13). We return to this below (see also Chapter III); however, it is useful already here to note that a reasonable approximation to the equation of state of stellar interiors is that of a perfect, fully ionized gas, according to which

$$p = \frac{k_{\mathrm{B}} \rho T}{\mu m_{\mathrm{u}}} , \tag{18}$$

where k_{B} is Boltzman's constant, μ is the mean molecular weight and m_{u} is the atomic mass unit. In this approximation, also, $\Gamma_1 = \Gamma_2 = \Gamma_3 = 5/3$.

2.2. EQUATIONS OF STELLAR STRUCTURE AND EVOLUTION

The equations presented in the previous section are completely general; thus in principle they can be used to describe simultaneously both the evolution of a star and its pulsations. In practice, however, this is not possible because of the huge range of time scales involved: evolution of the Sun, for example, takes place on a nuclear time scale of 10^9 y, while the pulsations are characterized by the dynamical time scale of order 1 h.

To circumvent this problem, stellar evolution is generally treated by assuming that the star is in hydrostatic equilibrium; then the time derivative in equation (7) is neglected. We furthermore neglect rotation, so that the velocity field vanishes; assuming also that gravity provides the only body force, we are left with [*cf.* equation (8)]

$$\boldsymbol{\nabla} p_0 = \rho_0 \mathbf{g}_0 = \rho_0 \boldsymbol{\nabla} \Phi_0 \,, \tag{19}$$

where we have denoted equilibrium quantities with the subscript "0." Poisson's equation (9) is unchanged; that is,

$$\nabla^2 \Phi_0 = -4\pi G \rho_0 \,. \tag{20}$$

If the star is spherically symmetric, it can be integrated once, to yield

$$g_0 = \frac{G}{r^2} \int_0^r 4\pi r'^2 \rho_0 \, \mathrm{d}r' = \frac{G m_{r,0}}{r^2} \,, \tag{21}$$

where $m_{r,0}$ is the mass contained in the sphere interior to r,

$$m_{r,0} = \int_0^r 4\pi r'^2 \rho_0 \, \mathrm{d}r' \,. \tag{22}$$

Thus equation (19) can also be written as

$$\frac{\mathrm{d}p_0}{\mathrm{d}r} = -g_0 \rho_0 = -\frac{G m_{r,0} \rho_0}{r^2} \,, \tag{23}$$

the familiar form of the equation of hydrostatic equilibrium.

The energy equation, equations (10) and (15), is typically rewritten as an equation for the energy flux. Assuming again spherical symmetry, we obtain

$$\frac{1}{r^2} \frac{\mathrm{d}}{\mathrm{d}r} (r^2 \mathcal{F}_{r,0}) = \rho \varepsilon_0 - \rho_0 \frac{\mathrm{d}E_0}{\mathrm{d}t} + \frac{p_0}{\rho_0} \frac{\mathrm{d}\rho_0}{\mathrm{d}t} \,, \tag{24}$$

or, with the luminosity $L_{r,0} = 4\pi r^2 \mathcal{F}_{r,0}$,

$$\frac{\mathrm{d}L_{r,0}}{\mathrm{d}r} = 4\pi r^2 \left(\rho \varepsilon_0 - \rho_0 \frac{\mathrm{d}E_0}{\mathrm{d}t} + \frac{p_0}{\rho_0} \frac{\mathrm{d}\rho_0}{\mathrm{d}t} \right) \,; \tag{25}$$

this is the commonly used form of the energy equation. It should be noticed that during 'normal' phases of evolution, where the energy production comes from quiet nuclear burning, the time-derivative terms in equation

(25) are small compared with the nuclear term, and hence are sometimes neglected.

Finally, the temperature gradient in the model must be determined from the requirements of energy transport. If the flux of energy is dominated by radiation, equation (16) directly relates the luminosity to the temperature gradient:

$$\frac{\mathrm{d}T_0}{\mathrm{d}r} = -\frac{3\kappa_0\rho_0}{16\pi r^2 a\tilde{c}T_0^3} L_{r,0} \,. \tag{26}$$

However, as noted in Section 2.1 convection may also have to be taken into account. The possibility of convective instability arises if the density decreases too slowly with increasing r; this may give rise to a situation corresponding effectively to having denser material above less-dense material. The circumstances giving rise to this are indicated by equation (26): if, for example, the opacity is very high, a large temperature gradient is required to transport the energy; since pressure, temperature and density are related by equation (18), and the pressure gradient is determined approximately by equation (23), a very steep temperature gradient may lead to a slow decrease of density with r. A more careful analysis (*e.g.* Kippenhahn & Weigert 1990) shows that the condition for *instability*, in terms of the gradient $\nabla = \mathrm{d}\ln T/\mathrm{d}\ln p$ of temperature with respect to pressure, is that

$$\nabla > \nabla_{\mathrm{ad}} \tag{27}$$

(the so-called Schwarzschild criterion), where

$$\nabla_{\mathrm{ad}} = \left(\frac{\partial \ln T}{\partial \ln p}\right)_{\mathrm{ad}} = \frac{\Gamma_2 - 1}{\Gamma_2} \tag{28}$$

[*cf.* equation (14)].

In regions of instability, energy transport is generally dominated by convection. The energy transport by this process depends on the size of the superadiabatic gradient $\nabla - \nabla_{\mathrm{ad}}$, as well as on the energy content of the gas which in turn is proportional to the density. In most of the interiors of stars, this process is so efficient that only a very small superadiabatic gradient is required to transport the entire stellar luminosity; thus $\nabla \simeq \nabla_{\mathrm{ad}}$, or

$$\frac{\mathrm{d}T_0}{\mathrm{d}r} \simeq -\nabla_{\mathrm{ad}} \frac{T_0}{p_0} \frac{G m_{r,0}\rho_0}{r^2} \,. \tag{29}$$

In these regions the structure of the star is essentially determined by the equation of state, the chemical composition and the (nearly constant) specific entropy. The latter is controlled by the matching to the neighbouring convectively stable regions, possibly through a boundary layer at the edge of the convection zone[1].

In stars with convective envelopes, the density is so small near the stellar surface that the temperature gradient becomes significantly superadiabatic; the detailed properties of this superadiabatic region controls the change in specific entropy between the atmosphere and the nearly adiabatic bulk of the convection zone. Here a more detailed description of convection is required, to relate the superadiabatic gradient to the luminosity and the properties of the gas. A typical example of the descriptions used in stellar modelling is the mixing-length treatment, in which the convective efficacy is parametrized by the mixing-length parameter $\alpha_{\rm c}$. By adjusting $\alpha_{\rm c}$ the superadiabatic gradient, and hence the specific entropy, may be changed; this in turn affects the overall structure of the model. In the solar case, $\alpha_{\rm c}$ is adjusted in order to obtain a model of the present Sun with the correct surface radius (*cf.* Chapter II, Section 1.2). Recently, detailed hydrodynamical modelling of the near-surface part of convection zones in the Sun and a few other stars has been carried out (*e.g.* Stein & Nordlund 1998; Trampedach *et al.* 1997; Ludwig, Freytag & Steffen 1999). Although the simulations extend only over the outer approximately 2,000 km of the convection zone, they are sufficiently deep that the lower part is approximately adiabatic; hence they essentially define the specific entropy and thereby the structure of the rest of the convection zone. Interestingly, in the solar case the resulting structure is largely consistent with calibrated models using the mixing-length treatment, as well as with the helioseismically determined depth of the solar convection zone (Rosenthal *et al.* 1999).

Stellar models are generally computed by following the change in structure as the star ages and the distribution of chemical composition changes. In the simplest form, only composition changes caused by nuclear reactions are taken into account. In that case the rate of change of, for example, the abundance by mass X of hydrogen can be written as

$$\frac{\mathrm{d}X}{\mathrm{d}t} = -r_X \; , \tag{30}$$

where r_X is the net destruction rate, obtained by summing over the reactions in which hydrogen takes part. Thus r_X is determined by the rates of individual reactions and the abundances of the relevant elements. It may be necessary, however, to take into account more extensive networks of nuclear reactions, not least in later stages of evolution. In convection zones matter is completely mixed on short time scales, while mixing due to other hydrodynamical processes may be important also in the convectively stable regions.

[1]For example, if convective overshoot must be taken into account; *e.g.* Zahn (1991)

It is increasingly becoming realized that diffusion and settling play important rôles in the chemical evolution of stars (for a review, see for example Michaud & Proffitt 1993). In the solar case, the effect is relatively modest but clearly observable as a result of the high precision of the helioseismic data (*e.g.* Cox, Guzik & Kidman 1989; Christensen-Dalsgaard, Proffitt & Thompson 1993; Richard *et al.* 1996). In somewhat hotter stars, with shallower outer convection zones, the effects are much more dramatic, potentially leading to drastic changes in the surface composition (*e.g.* Turcotte, Richer & Michaud 1998), as well as in the distribution of elements in the interior with potential consequences for the stability of the stars towards oscillations (Charpinet *et al.* 1997). On the other hand, the apparent absence of abundance anomalies in most stars indicates that settling is somehow suppressed, presumably by mixing processes outside the convection zone. This interplay between settling and mixing remains one of the most severe areas of uncertainty in current stellar modelling.

Although the equations of stellar structure and evolution, (22), (23), (25) and (26), may appear simple, their simplicity is misleading. To carry out the modelling, the equations must be supplemented with a description of the physical properties of matter in the stars. This includes the thermodynamical properties, already mentioned in the preceding section, as well as the opacity and the rates of nuclear reactions. To the level of detail required, for example, by the interpretation of the observed solar oscillation frequencies, the specification of these physical properties constitutes major and interesting research areas. Some aspects of these are discussed in Chapter III. It must be kept in mind also that the description of stellar evolution provided here is highly simplified, not least in the neglect of rotation and other hydrodynamical effects which might affect the structure of the star, either directly or through changes in the composition profile. Thus one of the goals of the studies of variable stars must be to test the limitations of this, often denoted 'standard', stellar evolution theory and infer where and how it should be improved.

2.3. LINEAR PERTURBATION ANALYSIS

2.3.1. *Linearized equations*

In many cases of stellar pulsation, including the solar oscillations, the amplitudes are so small that the pulsations can be described with high precision as small perturbations around the equilibrium structure obtained by the methods of the preceding section. Thus, for example, pressure is written as

$$p(\mathbf{r}, t) = p_0(\mathbf{r}) + p'(\mathbf{r}, t) , \tag{31}$$

where p' is a small perturbation. Here p' is the *Eulerian* perturbation, that is, the perturbation at a given spatial point. It is also convenient at times to use a description involving a reference frame following the motion. A perturbation in this frame is called a *Lagrangian* perturbation. If an element of gas is moved from $\mathbf{r}$ to $\mathbf{r} + \boldsymbol{\delta}\mathbf{r}$ due to the perturbation, the Lagrangian perturbation in pressure may be calculated as

$$\delta p(\mathbf{r}) = p(\mathbf{r} + \boldsymbol{\delta}\mathbf{r}) - p_0(\mathbf{r}) = p'(\mathbf{r}) + \boldsymbol{\delta}\mathbf{r} \cdot \boldsymbol{\nabla} p_0 \, . \tag{32}$$

Equation (32) is equivalent to the relation (2) between the local and the material time derivative. Note also that the velocity is given by the time derivative of the displacement $\boldsymbol{\delta}\mathbf{r}$,

$$\mathbf{v} = \frac{\partial \boldsymbol{\delta}\mathbf{r}}{\partial t} \, . \tag{33}$$

To obtain the lowest-order (linear) equations for the perturbations, we insert expressions such as equation (31) into the full equations, subtract equilibrium equations, and neglect quantities of order higher than one in p', ρ', $\mathbf{v}$, etc. For the continuity equation the result is

$$\frac{\partial \rho'}{\partial t} + \operatorname{div} (\rho_0 \mathbf{v}) = 0 \, , \tag{34}$$

or, by using equation (33) and integrating with respect to time,

$$\rho' + \operatorname{div} (\rho_0 \, \boldsymbol{\delta}\mathbf{r}) = 0 \, . \tag{35}$$

Note that this equation may also be written as [using the analogue to equation (32)]

$$\delta\rho + \rho_0 \operatorname{div} (\boldsymbol{\delta}\mathbf{r}) = 0 \, , \tag{36}$$

which corresponds to equation (4). The equations of motion become

$$\rho_0 \, \frac{\partial^2 \boldsymbol{\delta}\mathbf{r}}{\partial t^2} = \rho_0 \, \frac{\partial \mathbf{v}}{\partial t} = -\boldsymbol{\nabla} p' + \rho_0 \mathbf{g}' + \rho' \mathbf{g}_0 \, , \tag{37}$$

where, obviously, $\mathbf{g}' = \boldsymbol{\nabla}\Phi'$. The perturbation Φ' in the gravitational potential satisfies the perturbed Poisson equation

$$\nabla^2 \Phi' = -4\pi G \rho' \, . \tag{38}$$

To obtain the linearized energy equation we need to expand, for example, $\mathrm{d}p/\mathrm{d}t$. Using equation (2) yields, to first order,

$$\frac{\mathrm{d}p}{\mathrm{d}t} = \frac{\partial p}{\partial t} + \mathbf{v} \cdot \boldsymbol{\nabla} p = \frac{\partial p'}{\partial t} + \mathbf{v} \cdot \boldsymbol{\nabla} p_0 = \frac{\partial p'}{\partial t} + \frac{\partial \boldsymbol{\delta}\mathbf{r}}{\partial t} \cdot \boldsymbol{\nabla} p_0 = \frac{\partial \, \delta p}{\partial t} \, . \tag{39}$$

Note that to this order there is no difference between the local and the material time derivative of the *perturbations*. Thus we have for the energy equation, using, for example, equations (11) and (12),

$$\begin{aligned} \frac{\partial\,\delta Q}{\partial t} &= \frac{1}{\rho_0(\Gamma_{3,0}-1)}\left(\frac{\partial\,\delta p}{\partial t} - \frac{\Gamma_{1,0} p_0}{\rho_0}\frac{\partial\,\delta\rho}{\partial t}\right) \\ &= c_{P,0}\left(\frac{\partial \delta T}{\partial t} - \frac{\Gamma_{2,0}-1}{\Gamma_{2,0}}\frac{T_0}{p_0}\frac{\partial \delta p}{\partial t}\right) . \end{aligned} \tag{40}$$

This equation is most simply expressed in Lagrangian form, but it may be transformed into Eulerian form by using equation (32). From equation (15) the perturbation in the heating rate is given by

$$\rho_0 \frac{\partial\,\delta Q}{\partial t} = \delta(\rho\varepsilon - \operatorname{div} \boldsymbol{\mathcal{F}}) . \tag{41}$$

Here the perturbation in ε, assumed given as a function $\varepsilon(\rho, T, \{X_i\})$ of density, temperature and composition $\{X_i\}$, can be obtained as

$$\frac{\delta\varepsilon}{\varepsilon_0} = \left(\frac{\partial \ln\varepsilon}{\partial \ln\rho}\right)_{T,X_i}\frac{\delta\rho}{\rho_0} + \left(\frac{\partial \ln\varepsilon}{\partial \ln T}\right)_{\rho,X_i}\frac{\delta T}{T_0} , \tag{42}$$

since the Lagrangian perturbations in composition may be neglected. Also, it is relatively straightforward to obtain the perturbation to the radiative flux, in the diffusion approximation, from equation (16). On the other hand, the perturbation to the convective flux is highly uncertain, even within the simplified framework of the mixing-length treatment; although various time-dependent formulations exist (*e.g.* Unno 1967; Gough 1977; Balmforth 1992a) this remains a serious problem in any analysis of oscillations in stars with outer convection zones.

To simplify the notation, we drop the subscript "0" on equilibrium quantities from now on.

2.3.2. *The adiabatic approximation*

Under many circumstances the heating term can be neglected in equation (40). The resulting, so-called *adiabatic*, approximation greatly simplifies the treatment of stellar pulsation. To justify it, we consider the second form of equation (40), combined with equation (41), to obtain

$$\frac{\partial \delta T}{\partial t} - \frac{\Gamma_2 - 1}{\Gamma_2}\frac{T}{p}\frac{\partial \delta p}{\partial t} = \frac{1}{c_P}\frac{\partial\,\delta Q}{\partial t} = \frac{1}{c_P}\delta\left(\varepsilon - \frac{1}{\rho}\operatorname{div}\boldsymbol{\mathcal{F}}\right) ; \tag{43}$$

here we can compare the time derivative of δT with the dependence of the right-hand side on δT, concentrating on the term in $\operatorname{div}\boldsymbol{\mathcal{F}}$. From equation

(16), assuming that terms in the derivative of δT dominate, we obtain the estimate

$$\frac{1}{\rho c_P}\delta(\operatorname{div}\boldsymbol{\mathcal{F}}) \sim \frac{\delta T}{\tau_F} \;, \tag{44}$$

where

$$\tau_F = \frac{3\kappa\rho^2 c_P \ell^2}{4a\tilde{c}T^3} \simeq 10^{12}\,\frac{\kappa\rho^2\ell^2}{T^3} \;, \qquad \text{in cgs units} \;. \tag{45}$$

is a characteristic time scale for radiation over the characteristic length scale ℓ. The time derivative on the left-hand side of equation (43) may be estimated as $\delta T/\Pi$, where Π is the pulsation period. Thus, if $\tau_F \gg \Pi$, the right-hand side of equation (43) is much smaller than each term on the left-hand side and may often be neglected. However, it is evident that to study the excitation and damping of modes the energetics of the oscillations, as described by the nonadiabatic terms, must be included.

An estimate of τ_F for an entire main-sequence star such as the Sun yields $\tau_F \sim 10^7\,$y, corresponding approximately to the Kelvin-Helmholtz time for the star, which is a measure of the time required for the star to radiate its total thermal energy. This is enormously longer than the typical pulsation period of an hour, and hence in an average sense the adiabatic approximation is satisfied with very high accuracy. Even when estimated for small parts of the star, τ_F typically greatly exceeds the pulsation period. The only exception is in the very superficial layers of the star where the low density and small length scale reduce the thermal time scale. Here departures from adiabaticity become important. These regions have relatively small effect on the pulsation periods but they are crucial for the determination of the pulsation energetics (*cf.* Section 5).

For adiabatic motion we obtain from equation (40) that

$$\frac{\partial\,\delta p}{\partial t} - \frac{\Gamma_1 p}{\rho}\frac{\partial\,\delta\rho}{\partial t} = 0 \;, \tag{46}$$

or, by integrating over time,

$$\delta p = \frac{\Gamma_1 p}{\rho}\,\delta\rho \,. \tag{47}$$

In Eulerian form this becomes

$$p' + \boldsymbol{\delta}\mathbf{r}\cdot\boldsymbol{\nabla} p = \frac{\Gamma_1 p}{\rho}(\rho' + \boldsymbol{\delta}\mathbf{r}\cdot\boldsymbol{\nabla}\rho) \,. \tag{48}$$

2.4. EQUATIONS OF STELLAR PULSATION

We now concentrate on pulsations of stars that are assumed to have a spherically symmetric and time-independent equilibrium. This greatly simplifies the problem: the solution is separable in time, and in the angular coordinates (θ, ϕ) of the spherical polar coordinates (r, θ, ϕ) (where r is the distance to the centre, θ is co-latitude, *i.e.*, the angle from the polar axis, and ϕ is longitude). Then, time dependence is naturally expressed as a harmonic function, characterized by a frequency ω; since, furthermore, it simplifies the analysis to work in terms of complex variables, we express, for instance, the solution for the pressure perturbation as

$$p'(r, \theta, \phi, t) = \Re[\tilde{p}'(r) f(\theta, \phi) \exp(-i\omega t)] \; . \tag{49}$$

Here $f(\theta, \phi)$, which remains to be specified, describes the angular variation of the solution and, as indicated, the amplitude function $\tilde{p}'$ is a function of r alone. Note that the choice of sign of ω may be somewhat unconventional. The reason for the convention adopted here will become apparent later; the convention is set forth (not entirely seriously) in the form of a proposed resolution of Commission 27 of the International Astronomical Union (see Appendix).

Given a time dependence of this form, equations (37) can be written as

$$\omega^2 \boldsymbol{\delta}\mathbf{r} = \frac{1}{\rho} \boldsymbol{\nabla} p' - \mathbf{g}' - \frac{\rho'}{\rho} \mathbf{g} \; , \tag{50}$$

which has the form of a linear eigenvalue problem, ω^2 being the eigenvalue. Indeed, the right-hand side can be regarded as a linear operator $\mathcal{F}(\boldsymbol{\delta}\mathbf{r})$ on $\boldsymbol{\delta}\mathbf{r}$. This is most easily seen in the adiabatic approximation: here p' is related to ρ' by equation (48), and ρ', in turn, can be obtained from $\boldsymbol{\delta}\mathbf{r}$ from equation (35); also, given ρ', Φ' and hence $\mathbf{g}'$ can be obtained by integrating equation (38). We return to this formulation of the problem in Section 3.4, below.

To obtain the proper form of $f(\theta, \phi)$ in equation (49), we first express the displacement vector as

$$\boldsymbol{\delta}\mathbf{r} = \xi_r \mathbf{a}_r + \boldsymbol{\xi}_{\mathrm{h}} \; ,$$

where $\mathbf{a}_r$ is a unit vector in the radial direction, and $\boldsymbol{\xi}_{\mathrm{h}}$ is the tangential component of the displacement. We now take the tangential divergence $\mathrm{div}_{\mathrm{h}}$ of the equations of motion, and use the tangential part of the continuity equation to eliminate $\mathrm{div}_{\mathrm{h}} \, \boldsymbol{\xi}_{\mathrm{h}}$. In the resulting equation, derivatives with respect to θ and ϕ only occur in the combination ∇_{h}^2, where

$$\nabla_{\mathrm{h}}^2 = \frac{1}{r^2 \sin\theta} \frac{\partial}{\partial\theta} \sin\theta \frac{\partial}{\partial\theta} + \frac{1}{r^2 \sin^2\theta} \frac{\partial^2}{\partial\phi^2}$$

is the tangential part of the Laplace operator. The same is obviously true of Poisson's equation. Also, it may be shown that the energy equation, with the flux given by the diffusion approximation, results in the same behaviour.

This shows that separation in the angular variables can be achieved in terms of a function $f(\theta, \phi)$ which is an eigenfunction of $\nabla_{\rm h}^2$,

$$\nabla_{\rm h}^2 f = -\frac{1}{r^2} \Lambda f \ , \tag{51}$$

where Λ is a constant. A complete set of solutions to this eigenvalue problem are the spherical harmonics,

$$f(\theta, \phi) = (-1)^m c_{lm} P_l^m(\cos\theta) \exp(im\phi) \equiv Y_l^m(\theta, \phi) \ , \tag{52}$$

where P_l^m is a Legendre function and c_{lm} is a normalization constant, such that the integral of $|Y_l^m|^2$ over the unit sphere is unity. Here l and m are integers, such that $-l \leq m \leq l$ and $\Lambda = l(l+1)$.

With this separation of variables the pressure perturbation, for example, can be expressed as

$$p'(r, \theta, \phi, t) = \sqrt{4\pi} \Re[\tilde{p}'(r) Y_l^m(\theta, \phi) \exp(-i\omega t)] \ . \tag{53}$$

Also, it follows from the equations of motion that the displacement vector can be written as

$$\begin{aligned} \boldsymbol{\delta}\mathbf{r} &= \sqrt{4\pi} \Re \Bigg\{ \Bigg[\tilde{\xi}_r(r) Y_l^m(\theta, \phi) \mathbf{a}_r \\ &\quad + \frac{\tilde{\xi}_{\rm h}(r)}{L} \left(\frac{\partial Y_l^m}{\partial \theta} \mathbf{a}_\theta + \frac{1}{\sin\theta} \frac{\partial Y_l^m}{\partial \phi} \mathbf{a}_\phi \right) \Bigg] \exp(-i\omega t) \Bigg\} \ , \end{aligned} \tag{54}$$

where

$$\tilde{\xi}_{\rm h}(r) = \frac{L}{r\omega^2} \left(\frac{1}{\rho} \tilde{p}' - \tilde{\Phi}' \right) \ , \tag{55}$$

and $L = \sqrt{l(l+1)}$; in equation (54) $\mathbf{a}_\theta$ and $\mathbf{a}_\phi$ are unit vectors in the θ and ϕ directions. With this definition $\tilde{\xi}_r$ and $\tilde{\xi}_{\rm h}$ are essentially the root-mean-square radial and horizontal displacements.

It it instructive to notice, from equation (51), that

$$\frac{l(l+1)}{r^2} \simeq k_{\rm h}^2 \ , \tag{56}$$

where $k_{\rm h}$ is the tangential component of the wave number in a local approximation of the oscillation as a plane wave. Thus, for example, the horizontal surface wavelength of the mode is given by

$$\lambda_{\rm h} = \frac{2\pi}{k_{\rm h}} \simeq \frac{2\pi R}{\sqrt{l(l+1)}} \ ; \tag{57}$$

in other words, l is approximately the number of wavelengths around the stellar circumference. This identification is very useful in the asymptotic analysis of the oscillations. Also, it follows from, *e.g.*, equation (53) that m measures the number of nodes around the equator. A few examples of spherical harmonics are shown in Fig. 2. It should be noticed that with increasing degree the sectoral modes, with $m = \pm l$, become increasingly confined near the equator.

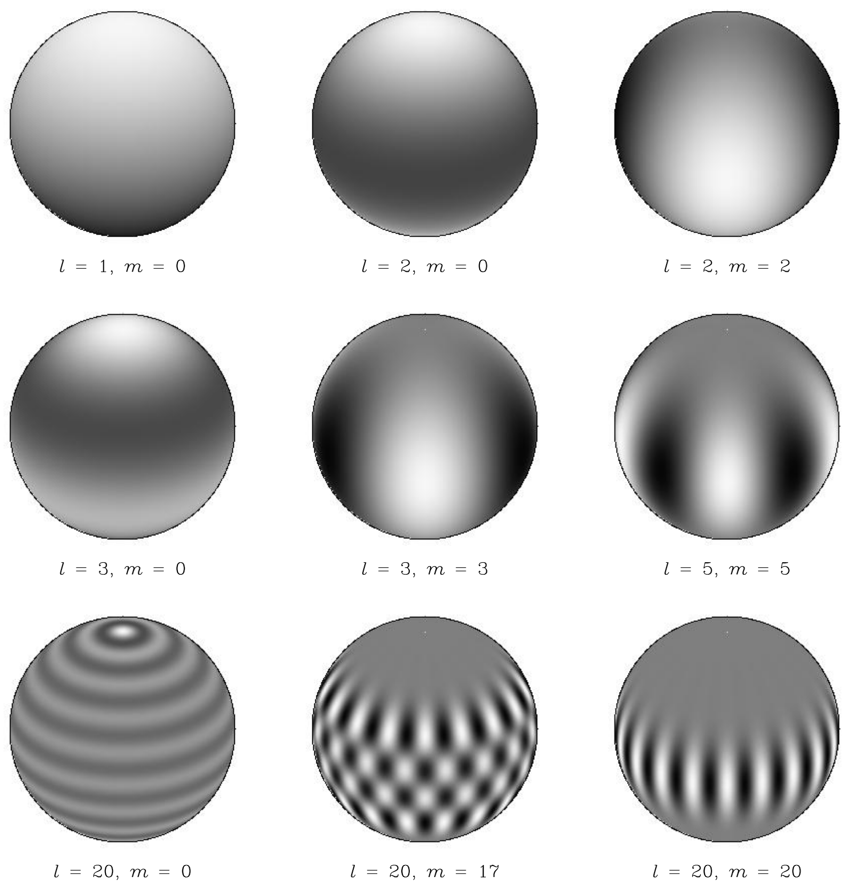

Figure 2. Examples of spherical harmonics, labelled by the degree l and azimuthal order m. For clarity the polar axis has been inclined 30° relative to the plane of the page.

Given the separation of variables, the equations of stellar pulsation are reduced to ordinary differential equations for the amplitude functions. For simplicity, we consider only the adiabatic case; writing the equations in terms of the variables $\{\xi_r,\ p',\ \Phi',\ \mathrm{d}\Phi'/\mathrm{d}r\}$ (where we have dropped the tildes) it is straightforward to obtain

$$\frac{\mathrm{d}\xi_r}{\mathrm{d}r} = -\left(\frac{2}{r} + \frac{1}{\Gamma_1 p}\frac{\mathrm{d}p}{\mathrm{d}r}\right)\xi_r + \frac{1}{\rho c^2}\left(\frac{S_l^2}{\omega^2} - 1\right)p' - \frac{l(l+1)}{\omega^2 r^2}\Phi' , \qquad (58)$$

$$\frac{\mathrm{d}p'}{\mathrm{d}r} = \rho(\omega^2 - N^2)\xi_r + \frac{1}{\Gamma_1 p}\frac{\mathrm{d}p}{\mathrm{d}r}p' + \rho\frac{\mathrm{d}\Phi'}{\mathrm{d}r} , \qquad (59)$$

and

$$\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}\Phi'}{\mathrm{d}r}\right) = -4\pi G\left(\frac{p'}{c^2} + \frac{\rho\xi_r}{g}N^2\right) + \frac{l(l+1)}{r^2}\Phi' . \qquad (60)$$

Here

$$c^2 = \frac{\Gamma_1 p}{\rho} \qquad (61)$$

is the squared adiabatic sound speed, and we have introduced the characteristic frequencies S_l and N (the so-called Lamb and buoyancy frequencies), defined by

$$S_l^2 = \frac{l(l+1)c^2}{r^2} = k_\mathrm{h}^2 c^2 , \qquad (62)$$

and

$$N^2 = g\left(\frac{1}{\Gamma_1 p}\frac{\mathrm{d}p}{\mathrm{d}r} - \frac{1}{\rho}\frac{\mathrm{d}\rho}{\mathrm{d}r}\right) . \qquad (63)$$

The physical meaning of these frequencies will become clearer in Section 3.1, below.

The equations must be combined with boundary conditions: two of these ensure regularity at the centre, $r = 0$, which is a regular singular point of the equations. One condition enforces continuity of Φ' and its gradient at the surface, $r = R$. Finally, the surface pressure perturbation must satisfy a dynamical condition. In its most simple form it imposes zero pressure perturbation on the perturbed surface, *i.e.*,

$$\delta p = 0 \qquad \text{at} \qquad r = R . \qquad (64)$$

The fourth-order system of differential equations (58) – (60), and the boundary conditions, define an eigenvalue problem that has solutions only for selected discrete values of ω. Thus for each (l, m) we obtain a set of eigenfrequencies ω_{nlm}, distinguished by their radial order n. The precise definition of n, for arbitrary stellar models, is a non-trivial problem (*e.g.* Lee 1985; Guenther 1991; Christensen-Dalsgaard & Mullan 1994). However, in many cases n is simply the number of radial nodes in, say, the radial component of the displacement, excluding a possible node at the centre. It may be shown that for the present case of adiabatic oscillation, ω^2 is always real; it follows that the eigenfunctions may be chosen to be real also.

It should be noticed that in the present case of a spherically symmetric star the frequencies are degenerate in azimuthal order: the definition of m is tied to the orientation of the coordinate system which, for a spherically symmetric star, can have no physical significance. Indeed, the equations and boundary conditions do not depend on m. As discussed in Section 4, this degeneracy is lifted by rotation.

The modes observed in several classes of pulsating stars, including the Sun, are either of high radial order or high degree. In such cases it is often possible, in approximate analyses, to make the so-called Cowling approximation, where the perturbation Φ' in the gravitational potential is neglected (Cowling 1941). This can be justified, at least partly, by noting that for modes of high order or high degree, and hence varying rapidly as a function of position, the contributions from regions where ρ' have opposite sign largely cancel in the solution to Poisson's equation (38). (On the other hand, Φ' should in general be included in numerical computations.) In this approximation, the order of the equations is reduced to two, greatly simplifying the analysis. In addition, equation (55) directly relates the surface pressure perturbation and horizontal displacement. Using also equations (64) and (32), we obtain

$$\frac{\xi_{\mathrm{h}}(R)}{\xi_r(R)} \simeq \frac{GM}{R^3} \frac{L}{\omega^2} , \tag{65}$$

where M is the total mass; from this it follows, for example, that at the observed solar frequencies and low or moderate degree the oscillations are predominantly in the radial direction.

3. Properties of adiabatic stellar pulsations

Here we discuss briefly some of the general properties of stellar pulsation. Although the computation of linear stellar oscillations is a relatively straightforward task, particularly in the adiabatic approximation, analytical techniques, not least the use of asymptotic analyses, have proved extremely useful in providing insight into the behaviour of the oscillations and the relation of their frequencies to stellar structure. Hence much of the discussion centres on the asymptotic behaviour of the oscillations.

From the point of view of helio- and asteroseismic investigations, it is important to realize which aspects of stellar structure are accessible to study, in the sense of having a direct effect on the oscillation frequencies. Within the adiabatic approximation it follows from equations (58) – (60) that the frequencies are completely determined by specifying p, ρ, g and Γ_1 as functions of the distance r to the centre. However, assuming that the equations of stellar structure are satisfied, p, g and ρ are related by equations (21) – (23). It follows that specifying just $\rho(r)$ and $\Gamma_1(r)$, say, completely determines the adiabatic oscillation frequencies. Conversely, the observed frequencies only provide direct information about these 'mechanical' quantities. To constrain other properties of the stellar interior, additional information has to be included, such as the equation of state or equations (25) and (26) determining the luminosity and temperature gradient (*e.g.* Gough & Kosovichev 1990). It is evident that the inferences obtained in such investigations may suffer from uncertainties in, for example, the assumed physics.

3.1. CHARACTERISTIC FREQUENCIES

3.1.1. *The dynamical frequency*

A characteristic time scale for dynamical changes to a star is provided by the free-fall time over a distance corresponding to a stellar radius, in the surface gravitational field of the star. The corresponding dynamical frequency,

$$\omega_{\rm dyn} = \left(\frac{GM}{R^3} \right)^{1/2} , \tag{66}$$

gives a measure of the oscillation frequencies of the star; thus the frequency of the fundamental radial mode is typically of order $\omega_{\rm dyn}$. It should be noticed that $\omega_{\rm dyn} \propto \bar\rho^{1/2}$ where $\bar\rho$ is the mean density of the star. For stellar models related by homology, the oscillation frequencies scale precisely as $\omega_{\rm dyn}$. This is sometimes expressed by introducing the pulsation constants

$$P_{nl} = \Pi_{nl} \left(\frac{\bar\rho}{\bar\rho_\odot} \right)^{1/2} \tag{67}$$

($\bar\rho_\odot$ being the mean density of the Sun), which are then the same for all homologously related stellar models. For realistic models P_{nl} depends somewhat on stellar properties; indeed, it is this often fairly subtle dependence which allows the use of observed oscillation frequencies to obtain information about the properties of stellar interiors.

3.1.2. *Properties of acoustic waves*

For many types of pulsating stars, including the Sun, the relevant oscillations are acoustic modes, often of fairly high order. In this case an asymptotic description can be obtained very simply, by approximating the modes locally by plane sound waves. They satisfy the dispersion relation

$$\omega^2 = c^2 |\mathbf{k}|^2 ,$$

where $\mathbf{k}$ is the wave vector. Thus the properties of the modes are entirely controlled by the variation of the adiabatic sound speed $c(r)$. It is instructive

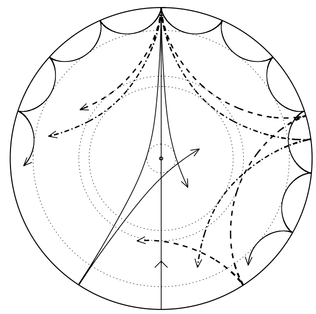

Figure 3. Propagation of rays of sound in a cross-section of the solar interior. The ray paths are bend by the increase in sound speed with depth until they reach *the inner turning point* (indicated by the dotted circles) where they undergo total internal refraction. At the surface the waves are reflected by the rapid decrease in density.

to note that, to the extent that the ideal gas law, equation (18), is satisfied,

$$c^2 \simeq \frac{\Gamma_1 k_{\rm B} T}{\mu m_{\rm u}} ; \tag{68}$$

hence the sound speed is essentially determined by T/μ.

To describe the radial variation of the mode, we separate $\mathbf{k}$ into radial and horizontal components k_r and $k_{\rm h}$ and use equation (56), to obtain

$$k_r^2 = \frac{\omega^2}{c^2} - \frac{L^2}{r^2} = \frac{\omega^2}{c^2}\left(1 - \frac{S_l^2}{\omega^2}\right) , \tag{69}$$

where again $L = \sqrt{l(l+1)}$ and S_l is defined in equation (62); thus this relation provides the physical meaning of S_l. This equation can be interpreted very simply in geometrical terms through the behaviour of rays of sound, as illustrated in Fig. 3. With increasing depth beneath the surface of a star temperature, and hence sound speed, increases. As a result, waves that are not propagating vertically are refracted, as indicated in equation (69) by the decrease in k_r with increasing c; the horizontal component $|\mathbf{k}_{\rm h}|$ of the wave vector, in contrast, increases with decreasing r. Thus the rays bend, as shown in Fig. 3. The waves travel horizontally at the lower turning point, $r = r_{\rm t}$, where $\omega = S_l$ and hence $k_r = 0$; thus, $r_{\rm t}$ is determined by

$$\frac{c(r_{\rm t})}{r_{\rm t}} = \frac{\omega}{L} . \tag{70}$$

For $r < r_{\rm t}$, k_r is imaginary and the wave decays exponentially. It follows from equation (70) that the lower turning point is located the closer to the centre, the lower is the degree or the higher is the frequency. Radial modes, with $l = 0$, penetrate the centre, whereas the modes of highest degree observed in the Sun, with $l \gtrsim 1000$, are trapped in the outer 0.2 % of the solar radius.

This ray description illustrates the behaviour of acoustic waves propagating through the star. The normal modes observed as global oscillations on the stellar surface arise through interference between such propagating waves. In particular, they share with the waves the total internal reflection at $r = r_{\rm t}$. In the case of the Sun, where modes of all degrees up to several thousand are observed, the oscillation frequencies of different modes thus reflect very different parts of the star; it is largely this variation in sensitivity which allows the detailed inversion for the properties of the solar interior as a function of position (see also Chapter II).

3.1.3. *An asymptotic description*

The above analysis provides a very simple example of the relation between the qualitative features of a mode and the properties of the relevant characteristic frequency (here the Lamb frequency S_l) in the stellar interior. Although this simple description is surprisingly successful in many cases, it ignores a number of important aspects. Near the surface, the scale heights of sound speed and density become small compared with the local wave length of the modes, invalidating the treatment in terms of locally plane sound waves. Furthermore, it ignores the rôle played by buoyancy as a restoring force of the oscillations.

A more complete description can be obtained from an asymptotic analysis of the pulsation equations, (58) – (60). This is most often carried out in the Cowling approximation neglecting the perturbation Φ' in the gravitational potential. Then the equations of adiabatic oscillation reduce to a second-order system, which can be treated by means of the JWKB method. A convenient way to formulate the problem was presented by Gough (*cf.* Deubner & Gough 1984), based on earlier analysis by Lamb (1932). Gough showed that, in terms of the quantity

$$\Psi = c^2 \rho^{1/2} {\rm div}\, \boldsymbol{\delta}\mathbf{r} \;, \tag{71}$$

the oscillation equations can be approximated by

$$\frac{{\rm d}^2\Psi}{{\rm d}r^2} = -K(r)\Psi \;, \tag{72}$$

where

$$K(r) = \frac{\omega^2}{c^2}\left[1 - \frac{\omega_{\rm c}^2}{\omega^2} - \frac{S_l^2}{\omega^2}\left(1 - \frac{N^2}{\omega^2}\right)\right] \;. \tag{73}$$

Here N^2 and S_l^2 were defined in equations (62) and (63), and

$$\omega_{\rm c}^2 = \frac{c^2}{4H^2}\left(1 - 2\frac{{\rm d}H}{{\rm d}r}\right) , \tag{74}$$

where $H = -({\rm d}\ln\rho/{\rm d}r)^{-1}$ is the density scale height.

In addition to the modes satisfying equation (72), there are modes for which div $\boldsymbol{\delta}\mathbf{r} \simeq 0$; these modes clearly cannot be analyzed in terms of Ψ. They approximately correspond to surface gravity waves, with frequencies satisfying

$$\omega^2 \simeq g k_{\rm h} , \tag{75}$$

and are usually known as *f modes*. We return to them in Section 3.2.3.

3.1.4. *Regions of mode trapping*

The physical meaning of equation (72) becomes clear if we make the identification $K = k_r^2$ where, as before, k_r is the radial component of the local wave number. Accordingly, a mode oscillates as a function of r in regions where $K > 0$; such regions are sometimes, a little imprecisely, referred to as regions of propagation, with reference to the waves of which the mode is made up. The mode is evanescent, decreasing or increasing exponentially, where $K < 0$. The detailed behaviour of the mode is thus controlled by the value of the frequency, relative to the characteristic frequencies S_l, N and $\omega_{\rm c}$. Note in particular that if $\omega^2 \gg \omega_{\rm c}^2, N^2$ we approximately recover equation (69) for k_r^2; this is the limit in which the approximation of the mode by plane sound waves is valid. More generally, the points where $K = 0$, marking the transition between the oscillatory and evanescent behaviour, are called turning points.

As a specific, but important, example Fig. 4 illustrates the characteristic frequencies in a model of the present Sun; the behaviour obtained in other main-sequence stars is qualitatively similar. It is evident that $\omega_{\rm c}$ is large only near the stellar surface, where the density scale height is small. Also, in most of the star $S_l \gg |N|$. It follows that, roughly speaking, a mode may have an oscillatory behaviour under two circumstances:

- **p)** $\omega > S_l, \omega > \omega_{\rm c}$;
- **g)** $\omega < N$.

Examples of propagating regions corresponding to these two cases are marked in Fig. 4. Modes corresponding to the former case are called *p modes*; it follows from the analysis given above that they are essentially standing sound waves, where the dominant restoring force is pressure. Modes corresponding to the latter cases are called *g modes*; here the dominant restoring force is buoyancy, and the modes have the character of standing internal gravity waves.

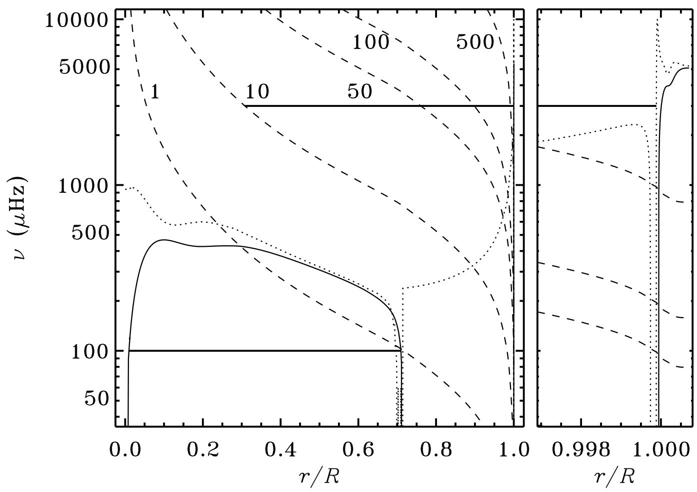

Figure 4. Characteristic frequencies $N/2\pi$ (solid line), $\omega_c/2\pi$ (dotted line) and S_l (dashed lines, labelled by l) for $l = 1$, 10, 50 and 500. The frequencies have been computed for Model S of Christensen-Dalsgaard *et al.* (1996). The heavy horizontal lines mark the trapping regions of a g mode of frequency 100 μHz and a p mode of frequency 3000 μHz and degree $l = 10$.

For the p modes, we may approximately neglect the term in N and, except near the surface, the term in ω_c. Thus we recover equation (69); in particular, the location of the lower turning point is approximately given by equation (70). Near the surface, on the other hand, $S_l \ll \omega$ and may be neglected; thus the location $r = R_t$ of the upper turning point is determined by $\omega \simeq \omega_c$. Physically, this corresponds to the reflection of the waves where the wavelength becomes comparable to the local density scale height. It should be noticed also from Fig. 4 that ω_c approximately tends to a constant in the stellar atmosphere. Indeed, since at least simple stellar atmosphere models are approximately isothermal, we here obtain that $H \simeq H_p$, the pressure scale height, which is furthermore approximately constant. Consequently,

$$\omega_c^2 \simeq \omega_a^2 = \frac{c^2}{4H_p^2} = \frac{\Gamma_1 \rho g^2}{4p} \simeq \frac{\Gamma_1 \mu m_u g_s^2}{4k_B T_s} \; , \qquad (76)$$

where ω_a is Lamb's acoustical cut-off frequency for an isothermal atmosphere (Lamb 1909); here g_s and T_s are the surface gravity and atmospheric temperature, and in the last approximation we used equation (18). Although ω_c displays a sharp peak (associated with the region of strong superadiabaticity near the top of the convection zone) rising substantially higher than this limiting atmospheric value, the trapping of the acoustic modes is dominated by the atmosphere. It follows that modes with frequencies exceeding $\omega_{\rm a}$ are only partially trapped; such modes loose energy in the form of running waves in the stellar atmosphere and hence may be expected to be rather strongly damped.

For g modes, particularly those of low frequency, the behaviour is dominated by the properties of the buoyancy frequency N; in particular, the turning points are approximately where $\omega = N$. The definition of N, equation (63), is intimately related to the condition for convective stability. Indeed, it may be shown that the proper dynamical condition for convective instability is the so-called Ledoux criterion:

$$N^2 < 0 \; , \qquad \text{or} \qquad \frac{\mathrm{d} \ln \rho}{\mathrm{d} \ln p} < \frac{1}{\Gamma_1} \; . \tag{77}$$

The relation to the Schwarzschild criterion in equation (27) becomes clear, if the approximation (18) is used to rewrite N^2 as

$$N^2 \simeq g^2 \frac{\rho}{p} (\nabla_{\rm ad} - \nabla + \nabla_\mu) \equiv N_0^2 + N_\mu^2 \; , \tag{78}$$

where $\nabla_\mu = \mathrm{d} \ln \mu / \mathrm{d} \ln p$, and we introduced

$$N_\mu^2 = -g \frac{\mathrm{d} \ln \mu}{\mathrm{d} r} = -4 \pi G \rho \frac{\mathrm{d} \ln \mu}{\mathrm{d} \ln m_r} \; . \tag{79}$$

If the chemical composition is homogeneous, the Ledoux and Schwarzschild criteria are clearly equivalent. In the presence of chemical inhomogeneities there is considerable uncertainty about which is the more appropriate criterion; under some circumstances the variation in μ may lead to overstable oscillations (often described as 'semiconvection'), which potentially mixes the material in convectively stable regions (*e.g.* Kippenhahn & Weigert 1990). For computational convenience, however, the Schwarzschild criterion is normally used. Since the general tendency of stellar evolution is to produce an accumulation of heavier elements near the centre, it leads to a growing positive ∇_μ and hence an increase in N^2. Indeed, this is visible in Fig. 4 in the peak close to the centre. More extreme examples of this will be discussed in Section 3.3.

3.2. ASYMPTOTIC PROPERTIES

From equation (72) we may obtain an approximate expression for the eigenfrequencies. Indeed, in terms of the simple description where locally the modes are plane waves, it is intuitively obvious that a standing wave requires the total change $\int k_r \mathrm{d}r$ between turning points to be an integral multiple of π. This condition can be made more precise through JWKB analysis of equation (72) (*e.g.* Fröman & Fröman 1965), which in particular deals with the behaviour close to the turning points. The result is that the modes satisfy

$$\omega \int_{r_1}^{r_2} \left[1 - \frac{\omega_{\mathrm{c}}^2}{\omega^2} - \frac{S_l^2}{\omega^2} \left(1 - \frac{N^2}{\omega^2} \right) \right]^{1/2} \frac{\mathrm{d}r}{c} \simeq \pi(n - 1/2) \, , \qquad (80)$$

where r_1 and r_2 are adjacent zeros of K such that $K > 0$ between them. This expression is valid in general, for both p modes and g modes. However, substantial simplifications can be achieved in each of the two cases, as discussed below.

For completeness we note that in some cases there may be more than one region in the star where $K > 0$, at a given frequency. In such cases the properties of the mode are dominated by that region where its amplitude is largest. However, this sometimes leads to *mixed modes*, with properties determined by two separate regions. Examples of such modes are discussed in Section 3.3.

3.2.1. *p modes*

For high frequencies we may assume that $|N^2/\omega^2| \ll 1$. Then equation (80) simplifies to

$$\omega \int_{r_1}^{r_2} \left[1 - \frac{\omega_{\mathrm{c}}^2}{\omega^2} - \frac{S_l^2}{\omega^2} \right]^{1/2} \frac{\mathrm{d}r}{c} \simeq \pi(n - 1/2) \, , \qquad (81)$$

where, as discussed above, $r_1 \simeq r_{\mathrm{t}}$ and $r_2 \simeq R_{\mathrm{t}}$. Further simplification results by noting that $\omega_{\mathrm{c}}/\omega \ll 1$ except near the upper turning point. As a result, it is possible to expand the integral to obtain

$$\omega \int_{r_{\mathrm{t}}}^{R} \left[1 - \frac{S_l^2}{\omega^2} \right]^{1/2} \frac{\mathrm{d}r}{c} \simeq \pi[n + \alpha(\omega)] \qquad (82)$$

(*e.g.* Christensen-Dalsgaard & Pérez Hernández 1992), where α, which as indicated in general depends on frequency, results from the expansion of the near-surface behaviour of ω_{c}. An expression of this form might in fact have been written down immediately on the basis of equation (69), by postulating an unknown phaseshift at the surface. It may also be written as

$$\frac{\pi(n + \alpha)}{\omega} \simeq F\left(\frac{\omega}{L} \right) \, , \qquad (83)$$

where

$$F(w) = \int_{r_t}^{R} \left(1 - \frac{c^2}{w^2 r^2}\right)^{1/2} \frac{\mathrm{d}r}{c} . \tag{84}$$

That the observed frequencies of solar oscillation satisfy the simple functional relation given by equation (83) was first found by Duvall (1982); this relation is therefore commonly known as *the Duvall law.*

For low-degree modes these relations may be simplified even further, by noting that in the integrand in equation (84) $(\ldots)^{1/2}$ differs from unity only close to the lower turning point which, for these modes, is situated very close to the centre. As a result it is possible to expand to integral to obtain, to lowest order, that

$$F(w) \simeq \int_0^R \frac{\mathrm{d}r}{c} - w^{-1}\frac{\pi}{2} ; \tag{85}$$

thus equation (83) may be approximated by

$$\omega = \frac{(n + L/2 + \alpha)\pi}{\displaystyle\int_0^R \frac{\mathrm{d}r}{c}} . \tag{86}$$

A more careful analysis shows that for low-degree modes L should be replaced by[2] $l + 1/2$ (*e.g.* Vandakurov 1967; Tassoul 1980). Thus we may write equation (86) as

$$\nu_{nl} \equiv \frac{\omega_{nl}}{2\pi} \simeq \left(n + \frac{l}{2} + \frac{1}{4} + \alpha\right)\Delta\nu , \tag{87}$$

where

$$\Delta\nu = \left[2\int_0^R \frac{\mathrm{d}r}{c}\right]^{-1} \tag{88}$$

is the inverse of twice the sound travel time between the centre and the surface. This equation predicts a uniform spacing $\Delta\nu$ in n of the frequencies of low-degree modes. Also, modes with the same value of $n + l/2$ should be almost degenerate,

$$\nu_{nl} \simeq \nu_{n-1\, l+2} . \tag{89}$$

This frequency pattern has been observed for the solar five-minute modes of low degree and may be used in the search for stellar oscillations of solar type.

[2] Note that, in any case, except at the lowest degrees this is an excellent approximation to the original definition of L; thus we shall use the two definitions interchangeably.

The *deviations* from the simple relation (87) have considerable diagnostic potential. The expansion of equation (84), leading to equation (86), can be extended to take into account the variation of c in the core (Gough 1986); alternatively it is possible to take the JWKB analysis of the oscillation equations to higher order (Tassoul 1980). As a result, one finds a departure from the approximate frequency coincidence obtained in equation (89),

$$d_{nl} \equiv \nu_{nl} - \nu_{n-1,l+2} \simeq -(4l+6)\frac{\Delta\nu}{4\pi^2\nu_{nl}}\int_0^R \frac{\mathrm{d}c}{\mathrm{d}r}\frac{\mathrm{d}r}{r} ; \tag{90}$$

here the integral is predominantly weighted towards the centre of the star, as a result of the factor r^{-1} in the integrand. This behaviour provides an important diagnostics of the structure of stellar cores. In particular, we note that, according to equation (68), the core sound speed is reduced as μ increases with the conversion of hydrogen to helium. As a result, d_{nl} is reduced, thus providing a measure of the evolutionary state of the star (*e.g.* Christensen-Dalsgaard 1984, 1988; Ulrich 1986; Gough & Novotny 1990).

It is often interesting to investigate the effects on the frequencies of small changes to the model. This is the case, in particular, for the Sun where helioseismic analyses indicate that the structure of solar models is already quite close to the true solar structure (*cf.* Chapter II). Such frequency changes may be estimated quite simply from the Duvall law. For simplicity, we consider two models of the same surface radius; the more general case can be addressed by first normalizing the frequencies with the dynamical frequency ω_{dyn} (*cf.* Section 3.1.1). We label the models with the superscripts (1) and (2), and introduce the differences $\delta\omega_{nl} = \omega_{nl}^{(2)} - \omega_{nl}^{(1)}$, $\delta_r c(r) = c^{(2)}(r) - c^{(1)}(r)$ and $\delta\alpha(\omega) = \alpha^{(2)}(\omega) - \alpha^{(1)}(\omega)$. By substituting $c^{(2)}(r) = c^{(1)}(r) + \delta_r c(r)$ and $\alpha^{(2)}(\omega) = \alpha^{(1)}(\omega) + \delta\alpha(\omega)$ into equation (83), retaining only terms linear in $\delta_r c$, $\delta\alpha$ and $\delta\omega$, one obtains

$$S_{nl}\frac{\delta\omega_{nl}}{\omega_{nl}} \simeq \int_{r_{\mathrm{t}}}^R \left(1 - \frac{L^2c^2}{r^2\omega_{nl}^2}\right)^{-1/2} \frac{\delta_r c}{c}\frac{\mathrm{d}r}{c} + \pi\frac{\delta\alpha(\omega_{nl})}{\omega_{nl}} , \tag{91}$$

where

$$S_{nl} = \int_{r_{\mathrm{t}}}^R \left(1 - \frac{L^2c^2}{r^2\omega_{nl}^2}\right)^{-1/2} \frac{\mathrm{d}r}{c} - \pi\frac{\mathrm{d}\alpha}{\mathrm{d}\omega} , \tag{92}$$

and we have suppressed the superscript (1). This relation was first derived by Christensen-Dalsgaard, Gough & Pérez Hernández (1988). Equation (91) may be written as

$$S_{nl}\frac{\delta\omega_{nl}}{\omega_{nl}} \simeq \mathcal{H}_1\left(\frac{\omega_{nl}}{L}\right) + \mathcal{H}_2(\omega_{nl}) , \tag{93}$$

where

$$\mathcal{H}_1(w) = \int_{r_t}^{R} \left(1 - \frac{c^2}{r^2 w^2}\right)^{-1/2} \frac{\delta_r c}{c} \frac{\mathrm{d}r}{c} , \tag{94}$$

and

$$\mathcal{H}_2(\omega) = \frac{\pi}{\omega} \delta\alpha(\omega) . \tag{95}$$

Some properties of this equation were discussed by Christensen-Dalsgaard, Gough & Thompson (1989), who pointed out that $\mathcal{H}_1(\omega/L)$ and $\mathcal{H}_2(\omega)$ can be obtained separately, to within a constant, by means of a double-spline fit of the expression (93) to p-mode frequency differences. The dependence of $\mathcal{H}_1$ on ω/L is determined by the sound-speed difference throughout the star; in fact, it is straightforward to verify that the contribution from $\mathcal{H}_1$ is essentially just an average of $\delta_r c/c$, weighted by the sound-travel time along the rays characterizing the mode. The contribution from $\mathcal{H}_2(\omega)$ depends on differences in the upper layers of the models.

3.2.2. *g modes*

For g modes in general $\omega^2 \ll S_l^2$, and we approximate equation (73) by

$$K(r) \simeq \frac{L^2}{r^2} \left(\frac{N^2}{\omega^2} - 1\right) . \tag{96}$$

The mode is assumed to be trapped between two zeros r_1 and r_2 of K, and hence, according to equation (80), the frequencies are determined by

$$L \int_{r_1}^{r_2} \left(\frac{N^2}{\omega^2} - 1\right)^{1/2} \frac{\mathrm{d}r}{r} = (n - 1/2)\pi . \tag{97}$$

We have here implicitly assumed that N has a single maximum, $N = N_{\max}$, so that at a given frequency the two turning points r_1 and r_2 are uniquely defined; otherwise, as discussed above, a more complex behaviour of the frequencies may result (see also Christensen-Dalsgaard, Dziembowski & Gough 1980). It is evident from equation (97) that for g modes $\omega < N_{\max}$. Also, for a given mode order n, $\omega \to N_{\max}$ as $l \to \infty$.

Figure 4 shows that in the Sun the region of g-mode trapping is located deep in the stellar interior; the same is true of other stars with extensive outer convection zones. Thus it is of interest to estimate whether the modes are likely to be visible on the stellar surface, given the extended intervening evanescent region. The analysis is particularly simple in the approximately adiabatically stratified convection zone, where $N^2 \simeq 0$. There, it follows from equations (72) and (73) that

$$\frac{\mathrm{d}^2\Psi}{\mathrm{d}r^2} \simeq \frac{l(l+1)}{r^2}\Psi , \tag{98}$$

with the solution

$$\Psi \propto r^{-l} \ . \tag{99}$$

Thus for low degree the decrease in the mode amplitude through the convection zone is modest. On the other hand, it is evident that higher-degree modes are efficiently trapped: since the radius at the base of the solar convection zone is approximately $r_{\rm cz} \simeq 0.7$, the decrease in amplitude for $l = 10$, say, is by a factor of about 35. Furthermore, modes with frequencies approaching $N_{\rm max}$ are trapped even more deeply and hence more efficiently. Hence one should probably not expect to see evidence for high-degree g modes on the surface of solar-like stars.

For high-order, low-degree g modes ω is much smaller than N over most of the interval $[r_1, r_2]$. This suggests that a similar approximation to the one leading to equation (86) should be possible. In fact, a proper asymptotic analysis (Tassoul 1980) shows that the frequencies of low-degree, high-order g modes are given by

$$\omega \simeq \frac{L \displaystyle\int_{r_1}^{r_2} N \frac{{\rm d}r}{r}}{\pi(n + l/2 + \alpha_g)} \ , \tag{100}$$

where α_g is a phase constant. Introducing the period $\Pi = 2\pi/\omega$, this may also be written as

$$\Pi \simeq \frac{\Pi_0}{L} \left(n + \frac{l}{2} + \alpha_g \right) \ , \tag{101}$$

where

$$\Pi_0 = \frac{2\pi^2}{\displaystyle\int_{r_1}^{r_2} N \frac{{\rm d}r}{r}} \ . \tag{102}$$

Thus in this case the *periods* are asymptotically equally spaced in the order of the mode. This behaviour is extremely important for the interpretation of pulsations of compact stars.

3.2.3. *f modes*

As noted in equation (75), a star also allows f modes with frequencies determined by

$$\omega^2 \simeq g_{\rm s} k_{\rm h} = L \frac{GM}{R^3} \ . \tag{103}$$

Thus, to this approximation, the frequencies scale precisely as $\omega_{\rm dyn}$ and hence depend only on the mean density of the star but not on its detailed internal structure. The modes have zero divergence of the displacement and hence, according to equations (36) and (47), $\delta\rho/\rho \simeq \delta p/p \simeq 0$; it may be shown that the displacement eigenfunction is exponential,

$$\xi_r \propto \exp(k_{\rm h} r) \ , \tag{104}$$

as is indeed found to be the case for surface gravity waves in deep water.

A more careful analysis must take into account the fact that gravity varies through the region over which the mode has substantial amplitude. The result is that the frequencies satisfy $\omega^2/g_s k_h = 1 - \epsilon(k_h)$, where

$$\epsilon = 2L^{-1} + 3\frac{\int (r-R)\rho \, \mathrm{e}^{2k_h r} \, \mathrm{d}r}{R \int \rho \, \mathrm{e}^{2k_h r} \, \mathrm{d}r} \tag{105}$$

(Gough 1993), which leads to a weak dependence of the frequencies on the density structure of the star. Some properties of this relation were discussed by Chitre, Christensen-Dalsgaard & Thompson (1998).

3.2.4. *Frequencies of a solar model*

To illustrate some of the asymptotic properties discussed here, it is instructive to consider numerically computed frequencies of adiabatic oscillation of a solar model, shown in Fig. 5. For clarity modes of given radial order n have been connected. It is immediately evident that the modes fall in two broad classes, corresponding to the p and the g modes; somewhat confusingly, at high degree the f-mode frequencies behave superficially rather like the frequencies of the p modes, despite their different physical nature. Also, the g modes clearly show the convergence towards the internal maximum $N_{\rm max}$ in the buoyancy frequency (the secondary accumulation of the g modes is related to a very weak secondary maximum in N, hardly visible in Fig. 4). Following common convention, we have assigned negative orders to the g modes, such that, at least in the limit of low frequency, $|n|$ is the number of radial nodes in the eigenfunction (*e.g.* Scuflaire 1974; Osaki 1975); also, the f modes are assigned $n = 0$. With this convention, frequency is an increasing function of n at given l, for all n.

3.3. MIXED MODES

In stars with convective cores the behaviour of the frequencies as the star ages may get rather complex, as a result of the variation in the buoyancy frequency N. This is illustrated in Figure 6 for the case of a $2.2 M_\odot$ evolution sequence. The convective core is fully mixed and here, therefore, the composition is uniform, with $\nabla_\mu = 0$ [*cf.* equations (78) and (79)]. However, in stars of this and higher masses the convective core generally shrinks during the evolution, leaving behind a steep gradient in the hydrogen abundance X, as shown in Figure 6a. This causes a sharp peak in ∇_μ and hence in N. When plotted as a function of mass fraction m/M, as in panel (b) of Figure 6, the location of this peak is essentially fixed

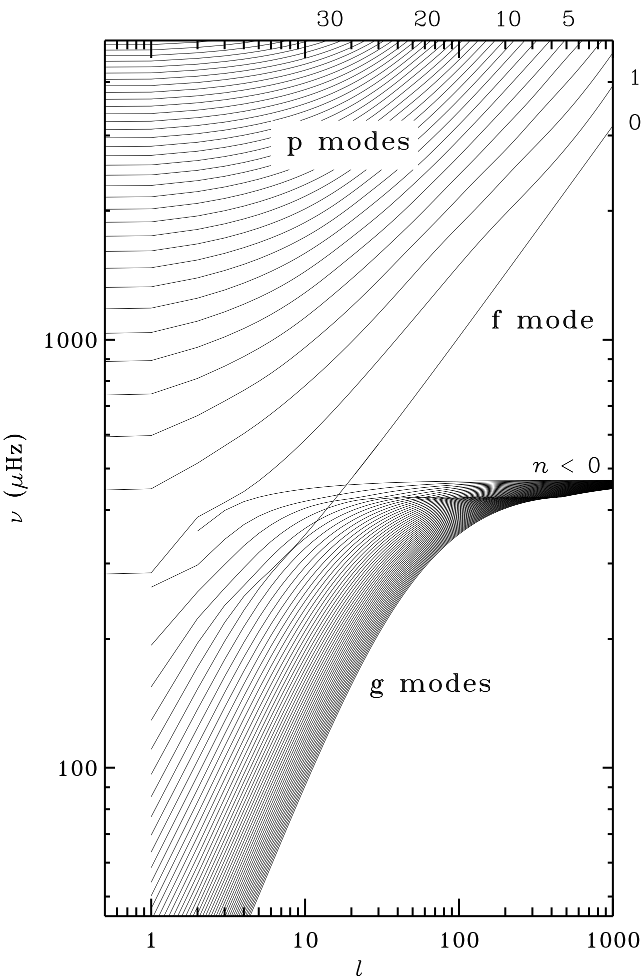

Figure 5. Cyclic frequencies $\nu = \omega/2\pi$, as functions of degree l, computed for a normal solar model. Selected values of the radial order n have been indicated.

although its width increases with the shrinking of the core[3]. However, as illustrated in Figure 6c, the location shifts towards smaller radius: this is a consequence of the increase with evolution of the central density and hence the decrease in the radial extent of a region of given mass. This also causes an increase in gravity g in this region and hence in N, visible in the fig-

[3]The erratic variation in N in the chemically inhomogeneous region is caused by small fluctuations, introduced by numerical errors, in $X(m)$.

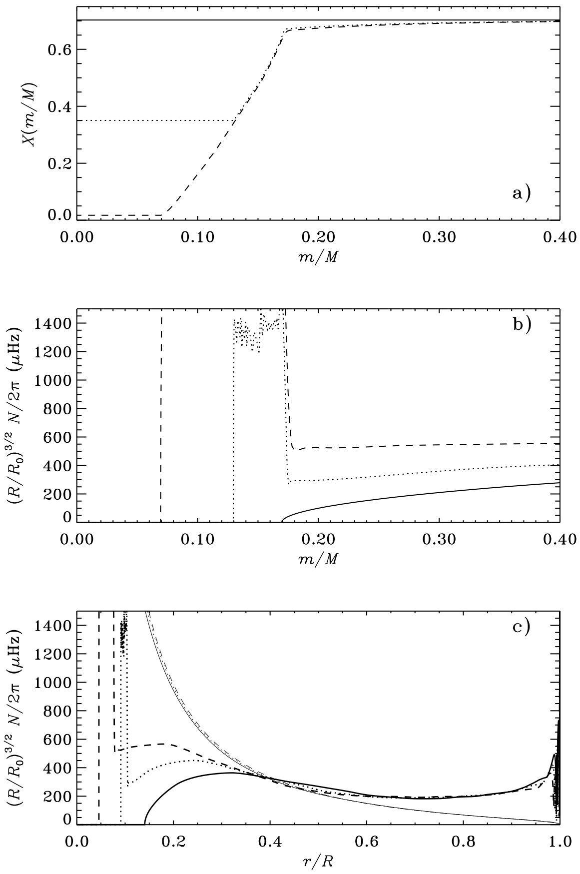

Figure 6. (a) Hydrogen content X against mass fraction m/M for three models in a $2.2M_\odot$ evolution sequence. The solid line is for age 0, the dotted line for age 0.47 Gyr and the dashed line for age 0.71 Gyr. Only the inner 40 per cent of the models is shown. (b) Scaled buoyancy frequency, expressed in terms of cyclic frequency, against m/M for the same three models. In the scaling factor, R and R_0 are the radii of the actual and the zero-age main sequence model, respectively. For the model of age 0.71 Gyr, the maximum value of $(R/R_0)^{3/2} N/2\pi$ is 2400 μHz. (c) Scaled buoyancy frequency N (heavy lines) and characteristic acoustic frequency S_l for $l = 2$ (thin lines), for the same three models, plotted against fractional radius r/R.

ure. To take out the essentially trivial homological variation as the stellar radius changes, the characteristic frequencies have been scaled by $R^{3/2}$ in Figure 6 (*cf.* equation 66): it is evident that S_l, and N in the outer parts of the model, are then largely independent of evolution. Thus the stellar envelope essentially changes homologously, while this is far from the case for the core; it follows that stellar oscillations sensitive to the structure of the core might be expected to show considerable variation with evolution.

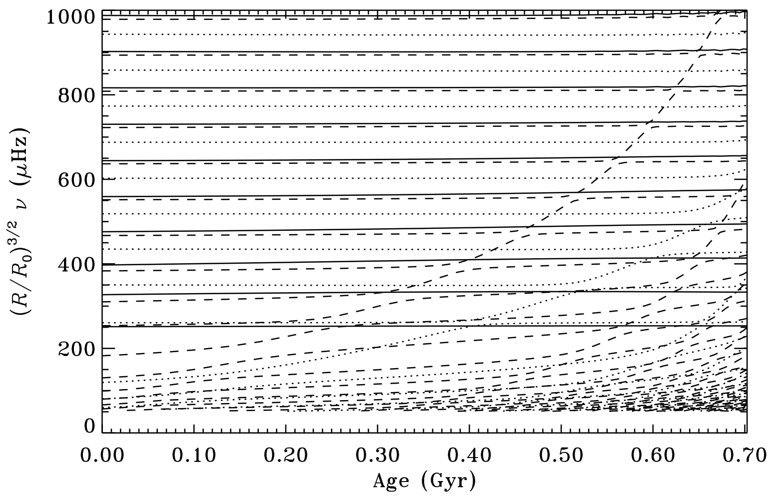

Figure 7. Oscillation frequencies, as functions of age, in a $2.2M_\odot$ evolution sequence. To eliminate the effect of the reduction in dynamical frequency with increasing radius R, the frequencies have been scaled by $(R/R_0)^3/2$, R_0 being the ZAMS radius of the model. Modes of the same radial order have been connected. The solid lines are for radial modes, of degree $l = 0$, the dotted lines are for $l = 1$ and the dashed lines for $l = 2$.

To illustrate the effects on the oscillation frequencies of these changes in the buoyancy frequency outside a convective core, Figure 7 shows the behaviour of the frequencies, as functions of stellar age, for a $2.2M_\odot$ evolution sequence. These models may represent δ Scuti stars; characteristic frequencies at a few ages in the sequence were illustrated in Figure 6. As in that figure I have applied the scaling according to $\omega_{\rm dyn}$. As a result, the frequencies of largely acoustic modes, including the radial modes, change very little with age. It should be noticed also that except at low order, the acoustic modes exhibit a distinct pattern, with a close pairing of the radial and $l = 2$ modes. Such a pattern of closely-spaced peaks is familiar from p-mode asymptotics (*cf.* equation 87).

The most striking feature of the computed frequencies, however, is the interaction for $l = 1$ and 2 between the p modes and the g modes. At zero age, there is a clear distinction between the p modes, with frequencies exceeding that of the lowest radial mode, and the g modes with frequencies below 200 μHz. However, with increasing age the scaled g-mode frequencies increase; this is a consequence of the increase in the scaled buoyancy

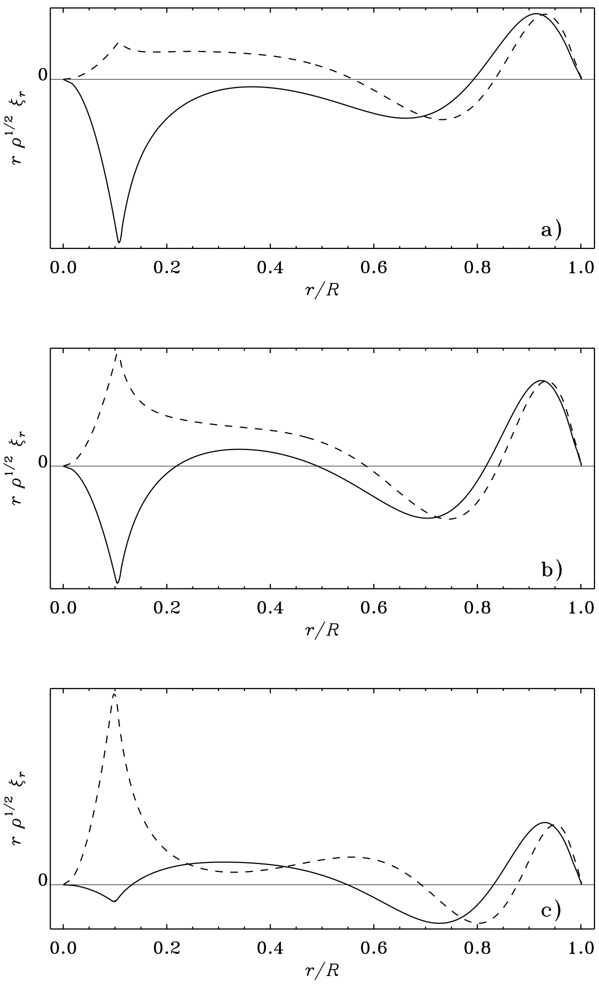

Figure 8. Scaled eigenfunctions for the $p_1(l = 2)$ mode (continuous line) and the $p_2(l = 2)$ mode (dashed line) in the vicinity of the avoided crossing near age 0.4 Gyr, $(R/R_0)^{3/2}\nu = 400\,\mu$Hz in Figure 7. (a) Age 0.36 Gyr. (b) Age 0.39 Gyr. (c) Age 0.44 Gyr.

frequency with age (*cf.* Figure 6c) which effectively acts to "pull up" the frequencies of the g modes. As was first found by Osaki (1975), this leads to an interaction between the p and g modes which takes place through a sequence of avoided crossings. At the avoided crossing the two modes exchange nature, while still maintaining the original labelling. Thus, for example, the $n = 1$ mode for $l = 2$, which at age zero is a purely acoustic mode of frequency 310 μHz takes on the nature of a g mode trapped just outside the convective core at the age 0.32 Gyr and at the age 0.4 Gyr again changes back to being predominantly an acoustic mode.

This behaviour is further illustrated by considering the eigenfunctions of these modes; examples of eigenfunctions near the $p_1 - p_2(l = 2)$ avoided crossing at age 0.4 Gyr are shown in Figure 8. Before the avoided crossing, the p_1 mode has a substantial amplitude near the edge of the convective core, and hence to a large extent behaves like a g mode, whereas the p_2 mode is predominantly a p mode, with largest amplitude in the outer parts. At the point of closest approach of the frequencies, at an age of 0.39 Gyr, both modes have a mixed character, with substantial amplitudes in the deep interior and near the surface, whereas after the avoided crossing the p_2 mode looks like a g mode, whereas the p_1 mode largely behaves like a p mode. It should be noted that this behaviour introduces a potential difference between the mathematical classification of the modes and their physical nature: modes with order $n > 0$, which in simple models would be p modes, may take on the character of g modes. Also, it is evident that the presence of the g-like modes in the p-mode spectrum, particularly at late evolutionary stages, complicates the analysis of observed frequencies. Dziembowski & Królikowska (1990) pointed out that mode selection might be affected by the larger energy, at given surface amplitude, of the modes that behave like g modes, thereby restricting the choice of modes in the identification. However, such arguments depend on the mechanisms responsible for exciting the modes and limiting their amplitudes, which are so far incompletely understood. It should also be noted that *if* g-mode like pulsations could in fact be identified, their frequencies would give strong constraints on conditions in the region just outside the stellar core. In fact, Dziembowski & Pamjatnykh (1991) pointed out that measurement of g-mode frequencies might provide a measure of the extent of convective overshoot from the core.

3.4. VARIATIONAL PRINCIPLE.

The formulation of the oscillation equations given in equation (50) is the starting point for powerful analyses of general properties of stellar pulsations. For convenience, we write it as

$$\omega^2 \boldsymbol{\delta}\mathbf{r} = \mathcal{F}(\boldsymbol{\delta}\mathbf{r}) \; , \tag{106}$$

where the right-hand side is the perturbed force per unit mass; as discussed in Section 2.4, this can be regarded as a linear operator on $\boldsymbol{\delta}\mathbf{r}$, as indicated.

The central result is that equation (106), applied to adiabatic oscillations, defines a *variational principle*. Specifically, by multiplying the equation by $\rho\boldsymbol{\delta}\mathbf{r}^*$ ("$*$" denoting the complex conjugate) and integrating over the volume V of the star, we obtain

$$\omega^2 = \frac{\int_V \boldsymbol{\delta}\mathbf{r}^* \cdot \mathcal{F}(\boldsymbol{\delta}\mathbf{r})\rho \mathrm{d}V}{\int_V |\boldsymbol{\delta}\mathbf{r}|^2 \rho \mathrm{d}V} . \tag{107}$$

We now consider adiabatic oscillations which satisfy the surface boundary condition given by equation (64). In this case it may be shown that the right-hand side of equation (107) is stationary with respect to small perturbations in the eigenfunction $\boldsymbol{\delta}\mathbf{r}$ (*e.g.* Chandrasekhar 1964). From a physical point of view, the assumptions ensure that the pulsating star is a conservative mechanical system; in particular, when $\delta p = 0$ there are no forces applied to the star from the outside. The stationarity then just reflects Hamilton's principle for a conservative system.

A very important application of this principle concerns the effect on the frequencies of perturbations to the equilibrium model or other aspects of the physics of the oscillations. Such perturbations can in general be expressed as a perturbation $\delta\mathcal{F}$ to the force in equation (106). It follows from the variational principle that their effect on the frequencies can be determined as

$$\delta\omega^2 = \frac{\int_V \boldsymbol{\delta}\mathbf{r}^* \cdot \delta\mathcal{F}(\boldsymbol{\delta}\mathbf{r})\rho \mathrm{d}V}{\int_V |\boldsymbol{\delta}\mathbf{r}|^2 \rho \mathrm{d}V} , \tag{108}$$

evaluated using the eigenfunction $\boldsymbol{\delta}\mathbf{r}$ of the unperturbed force operator. This equation is of course somewhat formal; specific applications of it will be discussed below, as well as in Chapter II.

4. Effects of rotation

So far, we have considered only oscillations of a spherically symmetric star; in this case, the frequencies are independent of the azimuthal order m. Departures from spherical symmetry lift this degeneracy, causing a frequency splitting according to m.

The most obvious, and most important, such departure is rotation. A simple description of the effects of rotation can be obtained by first noting that, according to equations (52) and (54), the oscillations depend on longitude ϕ and time t as $\cos(m\phi - \omega t)$, *i.e.*, as a wave running around the equator. We now consider a star rotating with angular velocity Ω and a mode of oscillation with frequency ω_0 in a frame rotating with the star; the coordinate system is chosen with polar axis along the axis of rotation. Letting ϕ' denote longitude in this frame, the oscillation therefore behaves as $\cos(m\phi' - \omega_0 t)$. The longitude ϕ in an inertial frame is related to ϕ' by $\phi' = \phi - \Omega t$ (*cf.* Fig. 9); consequently, the oscillation as observed from the

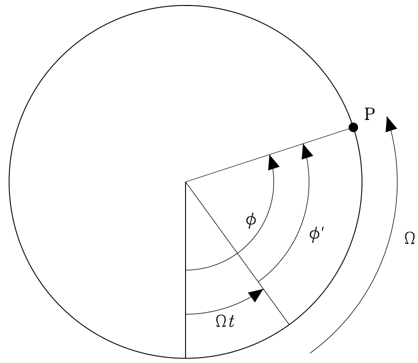

Figure 9. Geometry of rotational splitting, in a star rotating with angular velocity Ω. The point **P** has longitude ϕ' in the system rotating with the star and longitude $\phi = \phi' + \Omega t$ in the inertial system.

inertial frame depends on ϕ and t as

$$\cos(m\phi - m\Omega t - \omega_0 t) \equiv \cos(m\phi - \omega_m t) \ ,$$

where

$$\omega_m = \omega_0 + m\Omega \ . \tag{109}$$

Thus the frequencies are split according to m, the separation between adjacent values of m being simply the angular velocity; this is obviously just the result of the advection of the wave pattern with rotation. It should be noticed that, with the Çeşme sign convention introduced in equation (49) (*cf.* the Appendix), the frequency increases with increasing m.

4.1. EFFECTS OF SLOW ROTATION

This simple description contains the dominant physical effect, *i.e.*, advection, of rotation on the observed modes of oscillation, but it suffers from two problems: it assumes solid-body rotation, whereas the Sun, and presumably other stars, in fact rotate differentially; and it neglects the effects, such as the Coriolis and centrifugal forces, in the rotating frame. Differential rotation can be accounted for, roughly speaking, by replacing Ω in equation (109) by a suitable average; a natural expectation is that the appropriate weight is the energy density $\rho|\boldsymbol{\delta}\mathbf{r}|^2$ in the mode. To obtain a full description, including also the additional forces in the rotating frame, we must modify equation (50) appropriately. Here we consider only slowly rotating stars where the centrifugal force and other effects of second or higher order in Ω can be neglected; in particular, the distortion of the equilibrium structure caused by rotation is neglected. Then it is relatively simple to write down the required modification to equation 50); the result is, in an inertial frame,

$$\omega^2 \boldsymbol{\delta}\mathbf{r} = \frac{1}{\rho}\nabla p' - \mathbf{g}' - \frac{\rho'}{\rho}\mathbf{g} + 2m\omega\Omega\boldsymbol{\delta}\mathbf{r} - 2i\omega\boldsymbol{\Omega}\times\boldsymbol{\delta}\mathbf{r} \; , \qquad (110)$$

where $\boldsymbol{\Omega}$ is the rotation vector, of magnitude Ω and aligned with the rotation axis. The first term resulting from rotation is the $\mathcal{O}(\Omega)$ contribution from advection, as discussed above; this can be seen by combining it with the left-hand side to obtain, to $\mathcal{O}(\Omega)$, $(\omega - m\Omega)^2$. The last term is the Coriolis force (note that the velocity associated with the oscillation is $-i\omega\boldsymbol{\delta}\mathbf{r}$).

The terms arising from rotation obviously correspond to a perturbation to the force operator $\mathcal{F}$ in equation (106); hence the effect on the oscillation frequencies can be found from equation (108). The result can be written on the form

$$\omega_{nlm} = \omega_{nl0} + m\int_0^R\int_0^\pi K_{nlm}(r,\theta)\Omega(r,\theta) r\,\mathrm{d}r\,\mathrm{d}\theta \; , \qquad (111)$$

where the kernels K_{nlm} can be calculated from the eigenfunctions for the non-rotating model. It might be noted that the kernels depend only on m^2, so that the rotational splitting $\omega_{nlm} - \omega_{nl0}$ is an odd function of m. Also, the kernels are symmetrical around the equator (this follows immediately in the approximation where the kernels are determined by $\rho|\boldsymbol{\delta}\mathbf{r}|^2$); as a result, the rotational splitting is only sensitive to the component of Ω which is similarly symmetrical.

The general expression for the rotational kernels is quite complicated and will not be given here (*e.g.* Cuypers 1980; Gough 1981). It simplifies considerably in the case where $\Omega = \Omega(r)$ is assumed to be a function of r alone. The corresponding kernels do not depend on m, so that equation (111) predicts a uniform frequency splitting in m. This is often written on the form

$$\delta\omega_{nlm} \equiv \omega_{nlm} - \omega_{nl0} = m\beta_{nl}\int_0^R K_{nl}(r)\Omega(r)\mathrm{d}r \; , \qquad (112)$$

where

$$K_{nl} = \frac{(\xi_r^2 + \xi_\mathrm{h}^2 - 2L^{-1}\xi_r\xi_\mathrm{h} - L^{-2}\xi_\mathrm{h}^2)\, r^2\rho}{\int_0^R \left(\xi_r^2 + \xi_\mathrm{h}^2 - 2L^{-2}\xi_r\xi_\mathrm{h} - L^{-2}\xi_\mathrm{h}^2\right) r^2\rho\,\mathrm{d}r} \; , \qquad (113)$$

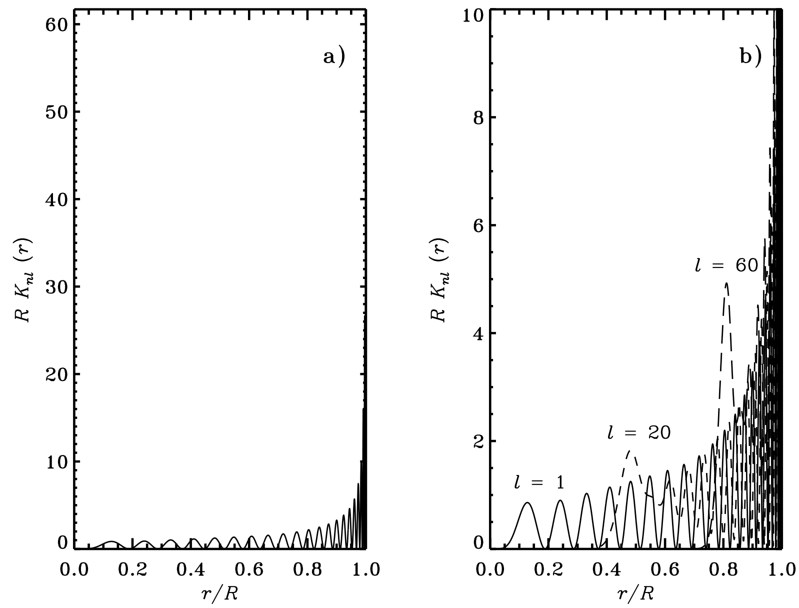

Figure 10. Kernels K_{nl} for the frequency splitting caused by spherically symmetric rotation (*cf.* eq. 113), for a model of the present Sun. In a) is plotted $RK_{nl}(r)$ for a mode with $l = 1$, $n = 22$ and $\nu = 3239\,\mu$Hz. The maximum value of $RK_{nl}(r)$ is 62. In b) is shown the same mode, on an expanded vertical scale, (continuous line) together with the modes $l = 20$, $n = 17$, $\nu = 3375\,\mu$Hz (short-dashed line), and $l = 60$, $n = 10$, $\nu = 3234\,\mu$Hz (long-dashed line). Notice that the kernels almost vanish inside the turning-point radius $r_{\rm t}$, and that there is an accumulation just outside the turning point.

and

$$\beta_{nl} = \frac{\displaystyle\int_0^R \left(\xi_r^2 + \xi_{\rm h}^2 - 2L^{-1}\xi_r\xi_{\rm h} - L^{-2}\xi_{\rm h}^2\right) r^2\rho{\rm d}r}{\displaystyle\int_0^R \left(\xi_r^2 + \xi_{\rm h}^2\right) r^2\rho{\rm d}r} \; ; \qquad (114)$$

here $L^2 = l(l+1)$. Examples of the kernels are shown in Fig. 10.

By construction K_{nl} is unimodular, *i.e.*, $\int K_{nl}(r){\rm d}r = 1$. Hence for uniform rotation, where $\Omega = \Omega_{\rm s}$ is constant,

$$\delta\omega_{nlm} = m\beta_{nl}\Omega_{\rm s} \; . \qquad (115)$$

In this case the effect of rotation is completely characterized by the constant β_{nl} [note that, following Ledoux (1951) β_{nl} is often written as $\beta_{nl} = 1 - C_{nl}$, where C_{nl} is the *Ledoux constant*].

For high-order or high-degree p modes the terms in ξ_r^2 and $\xi_{\rm h}^2$ dominate; β_{nl} is then close to one. Thus the rotational splitting between adjacent m-values is given approximately by the rotation rate. Physically, the neglected terms in equation (114) arise from the Coriolis force; thus rotational splitting for p modes is dominated by advection. For high-order g modes, on the other hand, we can neglect the terms containing ξ_r, so that

$$\beta_{nl} \simeq 1 - \frac{1}{L^2} \, . \tag{116}$$

In particular, the splitting of high-order g modes of degree 1 is only *half* the rotation rate.

4.2. EFFECTS OF MODERATE ROTATION ON STELLAR PULSATION

The perturbation formalism presented in the previous section is certainly sufficient for application to helioseismology and asteroseismology of white dwarfs. The Sun is a indeed a very slow rotator. We shall see later why, in spite of considerably faster rotation, the white dwarfs represent an easy case too. Amongst stars that may exhibit solar-like pulsation, rotation rates ten times the mean solar rate are not rare, while for the upper-main-sequence pulsators equatorial rotation rates fifty times solar must be regarded as typical. In these cases we have to go beyond the linear approximation in Ω

Soufi *et al.* (1998) developed a perturbation formalism leading to closed expressions for rotational frequency corrections, accurate to $\mathcal{O}(\Omega^3)$. The formalism is for any shellular ($\Omega = \Omega(r)$) rotation. Here we shall outline only the version for a uniform rotation. This is in fact sufficiently complicated. Earlier accurate treatments of the second-order effects rotation include papers by Chlebowski (1978), Saio (1981), Gough & Thompson (1990) and Dziembowski & Goode (1992).

4.2.1. *Effects on stellar structure*

Effects of the centrifugal force must now be included in the equilibrium structure. Thus, to the body force $\mathbf{f}$ in equation (8) we add

$$\mathbf{f}_{\rm cen} = \Omega^2 \varpi \mathbf{a}_\varpi \, , \tag{117}$$

where ϖ denotes the distance from rotation axis and $\mathbf{a}_\varpi$ the corresponding unit vector. In spherical coordinates we have

$$\mathbf{f}_{\rm cen} = \frac{\Omega^2}{3} \sin\theta(\sin\theta \mathbf{a}_r + \cos\theta \mathbf{a}_\theta) = \frac{\Omega^2}{3}\{(2r\mathbf{a}_r - \nabla[r^2 P_2(\cos\theta)]\} \, . \tag{118}$$

The latter form separates the centrifugal force in two parts. The first causes a modification of the mean radial structure of the star. The second part induces a distortion from sphericity described by the Legendre polynomial $P_2(\cos\theta)$. A similar separation holds for nonuniform rotation. However, in general the distorting part is not derived from a potential and, in the θ-dependent rotation rate, the distortion involves higher-order polynomials. In the case of the three-term rotation law such as used in the representation of the photospheric rotation of the Sun the distortion involves terms up to P_{10}.

Since our target accuracy is $\mathcal{O}(\Omega^3)$ we may treat the centrifugal force as a linear perturbation. Our small quantity is

$$\epsilon = \left(\frac{\Omega}{\omega_{\rm dyn}}\right)^2 . \tag{119}$$

We leave to the readers showing that, if the density does not decrease inward, then equation (119) implies that

$$\left(\frac{\Omega^2 r}{g_0}\right) \leq \epsilon . \tag{120}$$

Therefore the condition guarantees that the centrifugal force everywhere within the star is much smaller than the gravity force which justifies linearization of the equations for stellar structure about a spherically symmetric equilibrium. With the assumed accuracy, we may represent pressure on the form

$$p = p_0(r) + \epsilon[\tilde{p}_0(r) + \tilde{p}_2(r)P_2(\cos\theta)] . \tag{121}$$

In a similar manner we may represent density, ρ and the gravitational potential, Φ. If we use these representations in the modified equation (19), we get, collecting separately linear terms independent of θ and those proportional to P_2,

$$\frac{\mathrm{d}\tilde{p}_0}{\mathrm{d}r} + \tilde{\rho}_0\frac{\mathrm{d}\Phi_0}{\mathrm{d}r} + \rho_0\frac{\mathrm{d}\tilde{\Phi}_0}{\mathrm{d}r} = \frac{2r\rho\Omega^2}{3} , \tag{122}$$

$$\frac{\mathrm{d}\tilde{p}_2}{\mathrm{d}r} + \tilde{\rho}_2\frac{\mathrm{d}\Phi_0}{\mathrm{d}r} + \rho_0\frac{\mathrm{d}\tilde{\Phi}_2}{\mathrm{d}r} = \frac{-2r\rho\Omega^2}{3} , \tag{123}$$

and

$$\tilde{p}_2 = -\rho_0\left(\tilde{\Phi}_2 + \frac{\Omega^2 r^2}{3}\right) . \tag{124}$$

From Poisson's equation (8), which is already linear, we have

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}\tilde{\Phi}_s}{\mathrm{d}r}\right) - 6\tilde{\Phi}_s + 4\pi G r^2\tilde{\rho}_s = 0 \quad \text{for } s = 0 \text{ and } 2 . \tag{125}$$

Note that we have one equation less for the spherical part of the perturbation (subscript "0") than for the nonspherical part (subscript "2"). This fact has very important consequences. In the former case the system of equations is underdetermined and we have to consider the constraint of thermal balance to determine the modification to stellar structure. In the case of nonradial perturbation, on the other hand, the distortion is completely determined by equations (123) – (125). This may appear advantageous, but in fact it creates a serious problem, known as the *von Zeipel paradox* (von Zeipel 1924). Having determined $\tilde{\rho}_2$ and $\tilde{p}_2$, one may evaluate $\tilde{T}_2$ and, subsequently with the use of equation (16), the corresponding perturbation of $\mathcal{F}_{\rm rad}$. In general, the condition of thermal balance cannot be satisfied. Although the problem was recognized 75 years ago, we still do not have a fully satisfactory solution. We know that a fluid flow known as the *meridional circulation* arises but its role in material mixing and in determining the law of rotation, $\Omega(r, \theta)$, is still not fully understood.

Here, we assume that uniform rotation is an adequate approximation. The only rotational effect included in the mean, spherically symmetric, model is a modification of equation (23), which becomes

$$\frac{\mathrm{d}p_0}{\mathrm{d}r} = -(g_0 - \frac{2}{3}r\Omega^2)\rho_0 \ . \tag{126}$$

Thus the spherically symmetric part of the centrifugal perturbations has been absorbed in the mean structure of the model. This modification implies certain modifications of the normal mode frequencies. In order to evaluate the implied frequency shift we have to specify the nonrotating reference model; furthermore, we must remember that the centrifugal force has affected the past evolution of the star. We have various option. Here, following Soufi *et al.* (1998), we choose to compare rotating and nonrotating models of the same mass at the same effective temperature. To compute the evolution of the rotating models, we specify the total angular momentum and we assume that it is conserved during the evolution while Ω remains uniform.

To determine the nonradial distortion we solve equation (125) for $s = 2$ with

$$\tilde{\rho}_2 = \frac{1}{g_0}\frac{\mathrm{d}\rho_0}{\mathrm{d}r}\left(\tilde{\Phi}_2 + \frac{\Omega^2 r^2}{3}\right) \ , \tag{127}$$

which easily follows from equations (123) and (124). The boundary conditions are $\tilde{\Phi}_2 \propto r^2$ for $r \to 0$ and $\tilde{\Phi}_2 \propto r^{-3}$ for $r \to R$. It is not difficult to show that a unique solution always exists. Once we know $\tilde{\Phi}_2$, we may determine $\tilde{p}_2$ and $\tilde{\rho}_2$ with the help of equations (123) and (126), respectively.

4.2.2. *Effects on stellar oscillation frequencies and eigenfunctions*

The corrections to ω due to the Coriolis force and due to the nonradial part of the centrifugal force may now be evaluated by using the variational principle, equation (106). To determine the perturbation of the operator $\mathcal{F}$ we need – in addition to $\tilde{\Phi}_2$, $\tilde{p}_2$, and $\tilde{\rho}_2$ – also the perturbation of $\Gamma_1(p, \rho)$, which perhaps cannot be neglected. This may be obtained immediately from the equation of state, given $\tilde{p}_2$, and $\tilde{\rho}_2$. We write the operator in the following form:

$$\mathcal{F} = \mathcal{F}_0 + \epsilon \mathcal{F}_2 , \tag{128}$$

with

$$\mathcal{F}_0 = \mathcal{F}_{0,0} - 2i\omega \boldsymbol{\Omega} \times \boldsymbol{\delta}\mathbf{r} .$$

The operator $\mathcal{F}_{0,0}$ includes the spherically symmetric centrifugal perturbation. $\mathcal{F}_0$ in addition contains the Coriolis term, considered already in the previous section. The perturbed operator $\epsilon\mathcal{F}_2$ reflects the centrifugal distortion; it contains terms proportional to P_2 or $\mathrm{d}P_2/\mathrm{d}\theta$.

For the displacement vector, $\boldsymbol{\delta}\mathbf{r}$, the representation given in equation (54) is not sufficiently general, since the Coriolis force acting on the poloidal displacement generates a toroidal component. Thus we write

$$\boldsymbol{\delta}\mathbf{r} = \boldsymbol{\delta}\mathbf{r}_\mathrm{p} + \boldsymbol{\delta}\mathbf{r}_\mathrm{t} , \tag{129}$$

where $\boldsymbol{\delta}\mathbf{r}_\mathrm{p}$ is given by equation (54) and

$$\boldsymbol{\delta}\mathbf{r}_\mathrm{t} = \sqrt{4\pi}\frac{\xi_\mathrm{t}(r)}{L}\Re\left[\left(\frac{1}{\sin\theta}\frac{\partial Y_l^m}{\partial\phi}\mathbf{a}_\theta - \frac{\partial Y_l^m}{\partial\theta}\mathbf{a}_\phi\right)\exp(-i\omega t)\right] . \tag{130}$$

By definition we have

$$\mathrm{curl}_r\boldsymbol{\delta}\mathbf{r}_\mathrm{p} \equiv 0 , \quad \delta r_{\mathrm{t},r} \equiv 0 , \quad \mathrm{div}\boldsymbol{\delta}\mathbf{r}_\mathrm{t} \equiv 0 , \tag{131}$$

where curl_r is the radial component of the curl operator.

The poloidal components corresponding to a spherical harmonic Y_l^m generate toroidal components $\propto Y_{l\pm1}^m$. Taking curl_r of equation (54) we may obtain explicit formulae, accurate to $\mathcal{O}(\Omega^2)$, which express $\xi_{\mathrm{t},l\pm1}$ in terms of $\xi_{r,l}$ and $\xi_{\mathrm{h},l}$. These will not be reproduced here. We note only that although the toroidal components are of order Ω, their effect on the frequencies is of order $\mathcal{O}(\Omega^2) \propto \epsilon$.

In the perturbation method of Soufi *et al.* (1998) the basic poloidal eigenfunctions are obtained as solutions to the equation

$$\rho\omega^2\boldsymbol{\delta}\mathbf{r}_\mathrm{p} = \mathcal{F}_0\boldsymbol{\delta}\mathbf{r}_\mathrm{p} , \tag{132}$$

which is separable in terms of spherical harmonics. It is therefore almost as easy to solve as in the case of no rotation, except that now the operator depends on m. The solutions are eigenfunctions $\xi_{r,\mathrm{p}}$, $\xi_{\mathrm{h,p}}$ and eigenvalues ω accurate up to $\mathcal{O}(\Omega)$. Hence, ω includes the Ledoux-type linear frequency splitting, given by equation (115). It also contains some higher-order effects resulting from the Coriolis force.

The variational formula (108) is used to account for the centrifugal distortion and the feed-back effect of the toroidal components. We thus get

$$\delta\omega = \delta_{\mathrm{t}}\omega + \delta_{\mathrm{d}}\omega + \mathcal{O}(\Omega^4) , \tag{133}$$

where

$$\delta_{\mathrm{t}}\omega = \frac{1}{I}\int \boldsymbol{\delta}\mathbf{r}_{\mathrm{t}}^* \cdot \left(\frac{\omega}{2}\boldsymbol{\delta}\mathbf{r}_{\mathrm{t}} + \boldsymbol{\Omega}\times\boldsymbol{\delta}\mathbf{r}_{\mathrm{t}}\right)\rho \mathrm{d}V , \tag{134}$$

$$\delta_{\mathrm{d}}\omega = \frac{\epsilon}{2\omega I}\int \boldsymbol{\delta}\mathbf{r}_{\mathrm{p}}^* \cdot \mathcal{F}_2(\boldsymbol{\delta}\mathbf{r}_{\mathrm{p}})\rho \mathrm{d}V , \tag{135}$$

and

$$I = \int |\boldsymbol{\delta}\mathbf{r}_{\mathrm{p}}|^2 \rho \mathrm{d}V . \tag{136}$$

Note that $\boldsymbol{\delta}\mathbf{r}_{\mathrm{p}}$ and $\boldsymbol{\delta}\mathbf{r}_{\mathrm{t}}$ are calculated up to $\mathcal{O}(\Omega)$ and $\mathcal{O}(\Omega^2)$, respectively. Thus, the formulae include all cubic terms. One important reason for keeping these terms is that they affect inferences of the rotation rate. Let $\omega_{nlm}^{(\mathrm{o})}$ denote the frequency of the mode of radial order n, degree l and azimuthal order m, as measured in the observed system and let

$$\langle\Omega\rangle_{nlm} \equiv \frac{\omega_{nlm}^{(\mathrm{o})} - \omega_{nl\,-m}^{(\mathrm{o})}}{2m(1 - C_{nl})} \tag{137}$$

be the rotation rate inferred from this mode and his retrograde partner. The quadratic effects do not influence $\langle\Omega\rangle_{nlm}$ but the cubic effects do. Thus if Ω is sufficiently large $\langle\Omega\rangle_{nlm}$ would be different for different modes, indicating nonuniform rotation, even though the true rotation is uniform.

The explicit, general, expression for $\delta\omega$ is formidable and therefore we do not reproduce it here. However, there are relatively simple expressions in special cases, valid up to $\mathcal{O}(\Omega^2)$. In the case of radial pulsation we have an exact expression first found by Simon (1969),

$$\delta\omega = \delta_{\mathrm{t}}\omega = \frac{4\Omega^2}{3} . \tag{138}$$

For nonradial p modes there is a contribution of the same order from $\delta_{\mathrm{t}}\omega$. However, the total correction is dominated by $\delta_{\mathrm{d}}\omega$. The explicit formula for $\delta_{\mathrm{d}}\omega$ is quite complicated. However, starting from the general expression (*e.g.* Gough & Thompson 1990; Dziembowski & Goode 1992), and neglecting $\tilde{\Phi}_2$ in equations (124) and (127), which causes only a few per cent error, one gets the following simple asymptotic formula valid for $\omega \gg \omega_{\rm dyn}$:

$$\delta\omega \simeq \delta_{\rm d}\omega \simeq \frac{3\omega}{4} Q_{l,m} \epsilon \overline{\left(\frac{r}{R}\right)^3} , \tag{139}$$

where

$$Q_{l,m} = 2\pi \int_0^\pi {\rm d}\theta \sin\theta P_2 |Y_l^m|^2 = \frac{L^2 - 3m^2}{4L^2 - 3} , \tag{140}$$

and

$$\overline{\left(\frac{r}{R}\right)^3} = \frac{1}{I} \int \left(\frac{r}{R}\right)^3 |\delta r_{\rm p}|^2 \rho {\rm d}V , \tag{141}$$

I being defined by equation (136), is the average of the cube of the relative radius weighted with mode energy. This factor is typically in the range 0.3-0.4. To appreciate the magnitude of this second-order correction let us note that for solar-like p modes $(\omega/\omega_{\rm dyn})^2 \sim 10^3$ and $C_{n,l} \sim 1/n \sim 10^{-2}$ and, with the solar rotation rate, $\Omega/\omega \sim 10^{-4}$. Thus the second-order correction in the corotating system becomes bigger than the first-order effect. In the inertial system the linear term still dominates and the structure of the multiplets (n, l) remains very nearly equidistant and symmetric about the $m = 0$ component. (Note the m^2-dependence of the second-order correction as opposed to the m-dependence of the first-order correction.) However, already at a rotation rate five times faster than in the Sun the asymmetry is easily visible. At the rotation rates typical for the upper-main-sequence pulsators the simple linear structure of the multiplets is not recognizable even for low-order modes.

For g-modes, in the limit $\omega \ll L\omega_{\rm dyn}$ we have after Chlebowski (1978)

$$\delta\omega \simeq \delta_{\rm t}\omega \simeq -\frac{m^2\Omega^2}{\omega} \frac{4L^2(2L^2-3)-9}{2L^4(4L^2-3)} . \tag{142}$$

In oscillating white dwarfs $\Omega/\omega \sim 10^{-2}$ but at $l = 1$ and 2 the l-dependent factor is also $\sim 10^{-2}$; hence we get a correction which is much smaller than the linear one, equations (115) and (116).

4.2.3. *The case of near degeneracy*

The perturbation method described above is not applicable in the cases of the *near degeneracy* of the modes coupled by rotation. Let us consider two modes, say a and b, whose frequencies with corrections calculated according to equation (132) satisfy

$$\omega_{b,{\rm d}} \gg \omega_{a,{\rm d}} - \omega_{b,{\rm d}} \sim \Omega , \tag{143}$$

and such that

$$\lambda_{a,b} = \lambda_{b,a} = \int \boldsymbol{\delta}\mathbf{r}_a^* \cdot \mathcal{F}_2(\boldsymbol{\delta}\mathbf{r}_b)\mathrm{d}V \neq 0 \ . \tag{144}$$

In order to see when the last condition is satisfied we note that the angular part of $\lambda_{a,b}$ can be expressed in terms of

$$Q_{l_a,l_b,m_a,m_b} = \int \mathrm{d}\theta \sin\theta P_2 Y_{l_a}^{m_a *} Y_{l_b}^{m_b} \ . \tag{145}$$

It is easy to see that this does not vanish if

$$m_a = m_b \quad \text{and} \quad l_a = l_b \quad \text{or} \quad l_a = l_b \pm 2 \ . \tag{146}$$

The situation that equation (141) is satisfied and $l_a = l_b \pm 2$ is not rare. In fact, it may happen systematically for high-order p modes, as a consequence of the near-coincidence of the frequencies found asymptotically (cf eq. 89). With $l_a = l_b$, condition (141) may be occasionally satisfied at avoided crossings.

In the cases where equations (141) and (143) are valid we must use a perturbation formalism for the case of degeneracy; thus we must seek the solution of

$$\rho\omega^2\boldsymbol{\delta}\mathbf{r} = (\mathcal{F}_0 + \epsilon\mathcal{F}_2)(\boldsymbol{\delta}\mathbf{r}) \ , \tag{147}$$

on the form

$$\boldsymbol{\delta}\mathbf{r} = A_a\boldsymbol{\delta}\mathbf{r}_{a,\mathrm{d}} + A_b\boldsymbol{\delta}\mathbf{r}_{b,\mathrm{d}} \ . \tag{148}$$

The standard procedure, consisting in multiplying equation (146) by $\boldsymbol{\delta}\mathbf{r}_{a,\mathrm{d}}^*$ and by $\boldsymbol{\delta}\mathbf{r}_{b,\mathrm{d}}^*$ and integrating over V, leads to a homogeneous system for A_a and A_b. The condition for nonzero amplitudes yields two frequencies, ω_a and ω_a, which are given by the following expressions:

$$\omega_a^2 = \frac{S+D}{2} \quad \text{and} \quad \omega_b^2 = \frac{S-D}{2} \ , \tag{149}$$

where

$$S = \omega_{a,\mathrm{d}}^2 + \omega_{b,\mathrm{d}}^2 \ ,$$

and

$$D = \sqrt{(\omega_{a,\mathrm{d}}^2 - \omega_{b,\mathrm{d}}^2)^2 + \frac{4\lambda_{a,b}^2}{I_a I_b}} \ ,$$

if we choose

$$\omega_{a,\mathrm{d}} \leq \omega_{b,\mathrm{d}} \ .$$

For each eigenfrequency there are associated amplitude ratios

$$\left(\frac{A_b}{A_a}\right)_a = \frac{\lambda_{a,b}}{I_b(\omega_a^2 - \omega_{b,\mathrm{d}}^2)} \quad \text{and} \quad \left(\frac{A_a}{A_b}\right)_b = \frac{\lambda_{b,a}}{I_b(\omega_b^2 - \omega_{a,\mathrm{d}}^2)} \ . \tag{150}$$

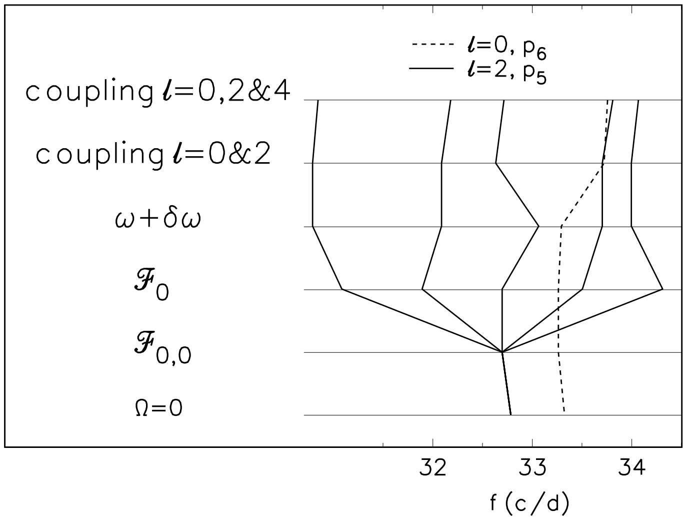

Figure 11. Effects of rotation, as seen from an inertial frame, on the frequencies of close $l = 0$ and 2 modes. At each step upward we add a new effect. At the lowest level we give frequencies in a model of a slightly evolved nonrotating $1.8 M_\odot$ star at $\log T_{\rm eff} = 3.876$. At subsequent levels we use a a model of a uniformly rotating star with the same parameters. At ZAMS the star had $V_{\rm rot} = 100$ km/s. The current value is 92 km/s. First, we see a small frequency decrease, similar for both modes, due to the modification of the radial structure. At the next level, we see mainly the frequency splitting by Ω. The effect of the Coriolis force is rather small for these p modes. The $\delta\omega$ term is nearly negligible for $l = 0$; however, it is quite large for $l = 2$, as a result of which the multiplet no longer has an equidistant appearance. Note that $\delta\omega$ pushes the $m = 0$ modes toward the $l = 0$ mode. This enhances the role of mode coupling which moves apart the interacting modes. This is the most significant effect of rotation on radial mode frequencies. Finally, we see that coupling with the components of a nearby $l = 4$ multiplet has a noticeable effect on frequencies of all six modes.

There may be more than two rotationally-coupled nearly degenerate modes. In particular, for higher-order p modes, a mode with $l = 4$ may satisfy the conditions with the $l = 0$ and $l = 2$ pair, hence entering into the interaction. There is an obvious generalization of the procedure described above to the case of three interacting modes.

Figure 11 illustrates various contributions to the rotational frequency perturbations in the case of close $l = 0$ and 2 modes. The model considered can be taken as representative for δ Scuti stars of luminosity class IV-V. The rotational velocity of 92 km/s is typical, if not somewhat low, for this type of object. The two modes are vibrationally unstable. Evidently the quadratic effects are very significant: the equidistant structure is not recognizable. We see also that the coupling between nearly degenerate modes has an important effect on mode frequencies. The cubic effect are not very large;

thus Ω as determined with the use of equation (137) agrees within a few per cent with the true value.

Rotational coupling of nearly degenerate modes causes mixed angular amplitude dependence for each of the modes. The relative contribution of the 'foreign' spherical harmonic in each of the coupled modes is determined by equation (150). This means, in particular, that there is no longer a pure radial mode if there is a close $l = 2$ mode. Chandrasekhar & Lebovitz (1962) invoked this effect to explain excitation of nonradial oscillations in β Cep stars. We now understand that nonradial modes are just as easily excitable as radial ones. However, rotational coupling remains an important and complicating effect which must be taken into account in connection with mode identification in observed oscillation spectra.

5. Damping and driving of stellar oscillations

As discussed in the introduction, stellar oscillations may be induced in two rather different manners: by being self-excited; or by being intrinsically damped but externally forced, typically by convection. Here we establish the basic framework for these processes.

5.1. SELF-EXCITED OSCILLATORS

Computation of the damping or excitation of linear modes requires treatment of the full nonadiabatic set of oscillation equations, including the energy equation as given by equations (40) and (41), with appropriate expressions for the perturbation in the energy-generation rate (eq. 42) and the flux divergence. The resulting equations have complex coefficients, leading to a complex eigenfrequency $\omega = \omega_{\rm r} + i\omega_{\rm i}$ where $\omega_{\rm r}$ and $\omega_{\rm i}$ are the real and imaginary parts of ω. Thus the time dependence of the oscillations is of the form

$$\cos(\omega_{\rm r} t)\exp(\omega_{\rm i} t) \ . \tag{151}$$

It follows that the mode is excited, or linearly unstable, when $\omega_{\rm i} > 0$, and damped otherwise [this rather convenient feature is a second consequence of our choice of sign convention in equation (49)].

Solution of the nonadiabatic equations directly leads to a determination of $\omega_{\rm i}$ and hence the stability of the mode. However, considerable more insight into the physics of the driving can be obtained from an integral expression for $\omega_{\rm i}$. This may be derived from the perturbation expression in equation (108) by noting that the full expression for the Eulerian pressure perturbation, including the nonadiabatic terms, is

$$\frac{p'}{p} = \Gamma_1 \frac{\rho'}{\rho} + \xi_r \left(\frac{{\rm d}\ln p}{{\rm d}r} - \Gamma_1 \frac{{\rm d}\ln \rho}{{\rm d}r} \right) + \frac{i}{\omega} \frac{\Gamma_3 - 1}{p} \delta(\rho\varepsilon - {\rm div}\, \boldsymbol{\mathcal{F}}) \ , \tag{152}$$

as is easily seen from equations (40) and (41). Here the first two terms on the right-hand side correspond to the adiabatic approximation, and hence are included in the definition of the operator $\mathcal{F}(\boldsymbol{\delta}\mathbf{r})$ in equation (106), whereas the last term, arising from the nonadiabatic effects, may be considered as a small perturbation. Substituting this into equation (108) and integrating by parts, we obtain $\delta\omega = i\omega_{\rm i}$, where

$$\omega_{\rm i} \simeq \frac{1}{2\omega^2} \frac{\displaystyle\int_V \frac{\delta\rho^*}{\rho}(\Gamma_3 - 1)\delta(\rho\varepsilon - \operatorname{div}\boldsymbol{\mathcal{F}}){\rm d}V}{\int_V \rho|\boldsymbol{\delta}\mathbf{r}|^2{\rm d}V} \tag{153}$$

(note that since the eigenfunctions used to evaluate the integral may be chosen to be real, so is the expression on the right-hand side of the equation).

Equation (153) has a very simple physical meaning: positive contributions come from those parts of the star that are heated at maximum compression, and where therefore $\delta\rho/\rho$ and $\delta(\rho\varepsilon - \operatorname{div}\boldsymbol{\mathcal{F}})$ have the same sign. This is precisely the condition for extracting mechanical energy from a Carnot heat engine. In fact, the term in $\delta(\rho\varepsilon)$ always contributes to the driving since compression, for a (nearly) adiabatic oscillation, leads to an increase in temperature and hence in the nuclear reaction rate. However, in most cases this contribution is relatively unimportant. Thus the stability of the star is decided by the, rather more complex, phase relations for the flux divergence. Instability may result if substantial 'bumps' in the opacity and its derivatives are located at the transition between nearly adiabatic, and strongly nonadiabatic, pulsation.

As derived here, equation (153) is often known as the *quasi-adiabatic approximation* to the excitation rate, which is estimated from the adiabatic eigenfunction. This approximation is often questionable, however: near the surface the true nonadiabatic eigenfunctions deviate strongly from the adiabatic eigenfunctions, and hence the variational property of equation (106) is no longer assured. Thus in practice reliable calculations of the excitation rate requires solution of the nonadiabatic equations. Even so, a generalized form of equation (153) may be derived and retains considerable value. In fact, it is easy to show, by multiplying equation (50) by $\rho\boldsymbol{\delta}\mathbf{r}^*$, integrating over the volume of the star and taking the imaginary part that, to leading order,

$$\omega_{\rm i} \simeq -\frac{1}{2\omega_{\rm r}} \frac{\Im\left[\displaystyle\int_V \frac{\delta\rho^*}{\rho}\delta p{\rm d}V\right]}{\int_V \rho|\boldsymbol{\delta}\mathbf{r}|^2{\rm d}V} \ . \tag{154}$$

By using again the nonadiabatic expression for δp in terms of $\delta\rho$ we essentially recover equation (153). When evaluated with the computed nonadiabatic eigenfunctions, this gives a very useful test of the accuracy of the excitation rate, obtained from the eigenfrequency; in addition, it shows which regions dominate the driving and damping of the mode.

Equation (154) also illustrates the fundamental importance of the relative phase of pressure and density for the excitation or damping of the mode. This is particularly significant when contributions other than the thermodynamic pressure need to be taken into account. The most important example is probably *turbulent pressure*, which may play an important rôle in the superficial parts of outer convection zones. Computation of the perturbation of the turbulent pressure requires a model of the time-dependent response of convection and hence is uncertain. On the basis of a time-dependent generalization of mixing-length theory (Gough 1976, 1977), Balmforth (1992a), Houdek (1996) and Houdek *et al.* (1999) found that the perturbation in turbulent pressure makes a substantial contribution to the stabilization of solar oscillations. A similar conclusion was reached by Nordlund & Stein (1998), on the basis of detailed hydrodynamical simulations of the interaction between convection and pulsations in a model of the solar convection zone.

5.2. STOCHASTIC EXCITATION

As mentioned above, stability calculations taking into account convection generally find that the solar oscillations are linearly stable. This motivates a search for driving mechanisms external to the oscillations, the most natural source being the very vigorous convection near the solar surface, where motion at near-sonic speed may be expected to be a strong source of acoustic waves (Lighthill 1952; Stein 1967). Similar excitation should then occur in other stars with near-surface convection. Since each mode feels the effect of a very large number of turbulent eddies, acting at random, the combined effect is that of a stochastic forcing of the mode.

Here we consider a very simple model of this process (Batchelor 1956; see also Christensen-Dalsgaard, Gough & Libbrecht 1989), consisting of a simple damped oscillator of amplitude $A(t)$, forced by a random function $f(t)$, and hence satisfying the equation[4]

$$\frac{\mathrm{d}^2 A}{\mathrm{d}t^2} + 2\eta \frac{\mathrm{d}A}{\mathrm{d}t} + \omega_0^2 A = f(t) \; ; \qquad (155)$$

here η is the linear damping rate, $\eta = -\omega_{\mathrm{i}}$. This equation is most easily dealt with in terms of its Fourier transform. We introduce the Fourier

[4]An equation of essentially this form may in fact be obtained from the full oscillation equations, including the convective forcing, by projecting onto the eigenmodes.

transforms $\tilde{A}(\omega)$ and $\tilde{f}(\omega)$ by

$$\tilde{A}(\omega) = \int A(t)e^{i\omega t}\mathrm{d}t \; , \qquad \tilde{f}(\omega) = \int f(t)e^{i\omega t}\mathrm{d}t \; , \tag{156}$$

where we do not attempt to specify the limits of integration precisely. According to equation (155) $\tilde{A}$ satisfies

$$-\omega^2\tilde{A} - 2i\eta\omega\tilde{A} + \omega_0^2\tilde{A} = \tilde{f} \; . \tag{157}$$

By solving this equation, we obtain the power spectrum of the oscillator as

$$P(\omega) = |\tilde{A}(\omega)|^2 = \frac{|\tilde{f}(\omega)|^2}{(\omega_0^2 - \omega^2)^2 + 4\eta^2\omega^2} \; . \tag{158}$$

In the vicinity of the peak in the spectrum, where $|\omega - \omega_0| \ll \omega_0$ the average power of the oscillation is therefore given by

$$\langle P(\omega)\rangle \simeq \frac{1}{4\omega_0^2}\frac{P_f(\omega)}{(\omega - \omega_0)^2 + \eta^2} \; , \tag{159}$$

where $P_f(\omega) = \langle|\tilde{f}(\omega)|^2\rangle$ is the average power of the forcing function.

Since $P_f(\omega)$ is often a slowly varying function of frequency, the frequency dependence of $\langle P(\omega)\rangle$ is dominated by the denominator in equation (159). The resulting profile is therefore approximately *Lorentzian*, with a width determined by the linear damping rate η. Consequently, under the assumption of stochastic excitation one can make a meaningful comparison between computed damping rates and observed line widths.

Kumar, Franklin & Goldreich (1988) made a careful statistical analysis of this problem, taking also into account the finite duration of the observations used to determine the mode parameters. In the limit where the observing time is short compared with the damping time η^{-1}, the power, or equivalently the mode energy E, is approximately exponentially distributed, with a distribution function

$$\mathcal{P}(E) = \exp(-E/\langle E\rangle) \; . \tag{160}$$

A similar distribution, although with significant deviations at high energy, was found by Chang & Gough (1998); strikingly, their results were very similar to those obtained from analysis of solar data by Chaplin *et al.* (1997). Additional observational support for this excitation mechanism was obtained very recently in the solar case by Gabriel *et al.* (1998) from analysis of the auto-correlation of the p-mode velocity signal obtained from the GOLF experiment on the SOHO spacecraft.

Excitation by turbulent convection has been modelled using descriptions of convection with varying degree of realism. In early calculations using simple mixing-length theory, Goldreich & Keeley (1977) found an approximate equipartition between the energy in convection and the resulting pulsations, the energy in one mode of oscillations being roughly equal to the energy in one convective eddy of a timescale corresponding to the period of the mode. Balmforth (1992b), Houdek (1996) and Houdek *et al.* (1999) obtained rather similar results using a time-dependent mixing-length description of convection; they showed that the resulting amplitudes were not inconsistent with those observed for the Sun. Furthermore, Nordlund & Stein (1998) showed that in detailed hydrodynamical modelling the energy input from convection was consistent with the energy requirements of the solar oscillations.

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Appendix: The Çeşme Resolution

In view of the fact

- that considerable and unnecessary confusion may arise from the lack of a common definition of the sign convention for the azimuthal order m in nonradial pulsation,
- that it is desirable to ensure that mode frequencies are increasing functions of m as well as of the degree l and the radial order n,

we hereby decree that the time dependence of nonradial pulsation, in terms of their angular frequency ω and time t, shall henceforth be expressed, on complex form, as

$$\exp(-\mathrm{i}\omega t) ,$$

such that m is positive for prograde modes, *i.e.*, modes travelling in the direction of rotation.

For Commission 27 of the International Astronomical Union,

D. W. Kurtz	J. Christensen-Dalsgaard
President	Vice-president

Given: Çeşme, on September 11, 1998.